Yangian invariance of the Tree Amplituhedron

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Scattering amplitudes

Scattering amplitudes are central objects in QFT. Interesting
- as an intermediate step to compute observables;
- as a means to gain insight into the formal structure of a specific model.

Traditional computational methods (Feynman diagrams)
- rapidly become inefficient;
- obscure properties of the theory.

New ideas: colour ordering, helicity amplitudes, new variables!
One example: tree-level gluon amplitudes in QCD

\[ 2g \rightarrow 3g \quad : \quad 25 \text{ diagrams} \]

\[ k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 \]
One example: tree-level gluon amplitudes in QCD

2g → 3g : 10 colour-ordered diagrams

\[ A^\text{tree}_5(1^\pm, 2^\pm, 3^\pm, 4^\pm, 5^\pm) = 0 \]

\[ A^\text{tree}_5(1^{\mp}, 2^\pm, 3^\pm, 4^\pm, 5^\pm) = 0 \]

\[ A^\text{tree}_5(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \]

\[ \langle ij \rangle = \epsilon_{\alpha\beta} \lambda^\alpha_i \lambda^\beta_j = \begin{vmatrix} \lambda^1_i & \lambda^1_j \\ \lambda^2_i & \lambda^2_j \end{vmatrix} \]
One example: tree-level gluon amplitudes in QCD

\[ 2g \rightarrow 3g : \quad 10 \text{ colour-ordered diagrams} \]

\[
A_5^{\text{tree}}(1^\pm, 2^\pm, 3^\pm, 4^\pm, 5^\pm) = 0
\]

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\]

\[
A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = \left(\frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle \langle 51\rangle}\right)
\]

\[
A_n^{\text{MHV}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \left(\frac{\langle ij\rangle^4}{\langle 12\rangle \langle 23\rangle \cdots \langle n1\rangle}\right)
\]

\[(N^k)\text{MHV} = (\text{next-to})^k\text{-maximally helicity-violating}\]  

[Parke, Taylor]
The simplest quantum field theory

Most symmetric theory in 4D is planar $\mathcal{N} = 4$ super Yang-Mills, a supersymmetric version of QCD.

- Maximal susy: spectrum is organized in a single supermultiplet with 2 gluons ($g^\pm$), 8 gluinos ($\psi^\pm$), 6 scalars ($\varphi$), all massless.

$$\Omega = g^+ + \eta^A \psi_A + \frac{1}{2} \eta^A \eta^B \varphi_{AB} + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \psi_D + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} g^-$$
Most symmetric theory in 4D is planar $\mathcal{N} = 4$ super Yang-Mills, a supersymmetric version of QCD.

- Maximal susy: spectrum is organized in a single supermultiplet with 2 gluons ($g^{\pm}$), 8 gluinos ($\psi^{\pm}$), 6 scalars ($\varphi$), all *massless*.

Possibility to compute superamplitudes

$$A_{n,k} = A_{n,k}(\Omega_1, \ldots, \Omega_n)$$

- (Ordinary + Dual) superconformal symmetries give rise to an infinite-dimensional **Yangian symmetry** $Y(\mathfrak{psu}(2,2|4))$ of tree-level scattering amplitudes.
The power of momentum supertwistors

“Masslessness” of the spectrum + conformal symmetry → introduce momentum supertwistors for describing the kinematics.

Instead of four-momenta $p^\mu$ with $\mu = 0, 1, 2, 3$ and Grassmann-odd $\eta^A$ with $A = 1, 2, 3, 4$

use mom. supertwistors $Z^A$ with $A = \left( \alpha, \dot{\alpha} = 0, 1, \dot{0}, \dot{1} \right)$

The $Z^A_i$ enforce massless momenta and momentum conservation.

Generating function for every tree-level $\mathcal{N} = 4$ SYM amplitude

$$\mathcal{L}_{n,k} = \frac{1}{\text{GL}(k)} \int \frac{d^{k \times n} c_{\alpha i}}{(1 \cdots k)(2 \cdots k + 1) \cdots (n \cdots k - 1)} \delta^{4|4}(C \cdot Z)$$

[(Arkani-Hamed, Cachazo, Cheung, Kaplan) (Mason, Skinner)]
Towards the amplituhedron: motivation

Two remarkable results inspired the amplituhedron:

\[ \mathcal{L}_{n,k} = \frac{1}{\text{GL}(k)} \int \frac{\text{d}^{k \times n} c_{\alpha i}}{(1 \cdots k)(2 \cdots k + 1) \cdots (n \cdots k - 1)} \delta^{4|4}(C \cdot Z) \]

Residues of \( \mathcal{L}_{n,k} \) \( \longleftrightarrow \) cells of positive Grassmannian \( G_+(k, n) \)
permutations of \( S_n \).

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

NMHV tree-level amplitudes can be thought of as volumes of polytopes in twistor space.

[Hodges]
Towards the amplituhedron: positive means inside

Simplex in $\mathbb{RP}^{n-1}$

Interior of a simplex

$$Y^A = \sum_i c_i Z_i^A , \quad c_i > 0$$

Points inside are described by the positive $n$-tuple

$$(c_1 \ c_2 \ldots \ c_n)/\text{GL}(1), \quad \text{a point in } G_+(1, n).$$
Towards the amplituhedron: positive also means convex

Polytope in $\mathbb{RP}^m$

Interior only canonically defined if the polytope is convex, i.e.

\[
Z = \begin{pmatrix}
Z_1^1 & Z_2^1 & \cdots & Z_n^1 \\
\vdots & \vdots & \ddots & \vdots \\
Z_1^{1+m} & Z_2^{1+m} & \cdots & Z_n^{1+m}
\end{pmatrix} \in M_+(1+m, n)
\]
### Tree-level amplituhedron

**Interior of an $n$-polytope in $\mathbb{RP}^m$**

\[
\mathcal{A}^{\text{tree}}_{n,1;m}[Z] = \left\{ Y^A = \sum_i c_i Z_i^A , \quad C = (c_1 \ldots c_n) \in G_+(1,n) \quad Z = (Z_1 \ldots Z_n) \in M_+(1+m,n) \right\}
\]
Tree-level amplituhedron

Interior of an \( n \)-polytope in \( \mathbb{RP}^m \)

\[
\mathcal{A}_{\text{tree}}^{n,1;m}[Z] = \left\{ Y^A = \sum_i c_i Z_i^A, \quad C = (c_1 \ldots c_n) \in G_+(1, n), \quad Z = (Z_1 \ldots Z_n) \in M_+(1 + m, n) \right\}
\]

Generalize this picture to account for \( N^k \text{MHV} \) amplitudes

Tree-level amplituhedron

\[
\mathcal{A}_{\text{tree}}^{n,k;m}[Z] = \left\{ Y \in G(k, k+m) : Y = C \cdot Z, \quad C \in G_+(k, n), \quad Z \in M_+(k + m, n) \right\}
\]

[Arkani-Hamed, Trnka]
Volume forms and volume functions

**Volume form**

Top-dimensional differential form $\Omega_{n,k}^{(m)}$ defined on $\mathcal{A}_{n,k;m}^{\text{tree}}$ with only logarithmic singularities on its boundaries.

- (top-dimensional) $\forall Y \in G(k, k + m) \rightarrow \Omega_{n,k}^{(m)}$ is an $mk$-form
- (log-singularity) approaching any boundary, $\Omega_{n,k}^{(m)} \sim \frac{d\alpha}{\alpha}$
Volume forms and volume functions

Volume form

Top-dimensional differential form $\Omega_{n,k}^{(m)}$ defined on $\mathcal{A}_{n,k;m}^{\text{tree}}$ with only logarithmic singularities on its boundaries.

top-dimensional: $Y \in G(k, k + m) \rightarrow \Omega_{n,k}^{(m)}$ is an $mk$-form
log-singularity: approaching any boundary, $\Omega_{n,k}^{(m)} \sim \frac{d\alpha}{\alpha}$

If $Y = \alpha_1 Z_1 + \alpha_2 Z_2 + Z_3$,

$$\Omega_{3,1}^{(2)} = \frac{d\alpha_1}{\alpha_1} \wedge \frac{d\alpha_2}{\alpha_2} = \frac{1}{2} \left( \frac{\langle 123 \rangle^2}{\langle Y \rangle} \right) \frac{\langle Yd^2Y \rangle}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle}.$$

volume function $\Omega_{3,1}^{(2)}$
Get to the amplitude

\[
\frac{A_{n,k}}{A_{n,0}} = \int_{\mathcal{A}^{\text{tree}}_{n,k;m}} \Omega^{(m)}_{n,k}(Y, Z) = \int d^{m \cdot k} \phi \Omega^{(m)}_{n,k}(Y^*, Z)
\]

"Scattering amplitudes are volumes of (dual) amplituhedra"

The physics, i.e. the kinematics of scattering particles, is encoded in $Z^A$ variables, bosonized version of momentum supertwistors $Z^A$:

\[
Z_i^A = \begin{pmatrix}
\lambda_i^\alpha \\
\tilde{\mu}_i^{\dot{\alpha}} \\
\phi_1 \cdot \chi_i \\
\vdots \\
\phi_k \cdot \chi_i
\end{pmatrix}, \quad \text{with} \quad \begin{pmatrix}
\lambda_i^\alpha \\
\tilde{\mu}_i^{\dot{\alpha}} \\
\chi_i^A \\
\phi_\alpha^A
\end{pmatrix}
\]

are bosonic d.o.f. of $Z^A$

are fermionic d.o.f. of $Z^A$

are auxiliary fermionic d.o.f.
1. Scattering amplitudes in planar $\mathcal{N} = 4$ SYM
   - Uncovering the simplicity of amplitudes
   - $\mathcal{N} = 4$ super Yang-Mills
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2. The Amplituhedron as a spin chain
   - Algebraic Bethe Ansatz: $XXX_{1/2}$ spin chain
   - The Amplituhedron story
XXX\textsubscript{1/2} spin chain: the setup

- Unidimensional model with periodic boundary conditions;
- \( N \) spin sites, each one acted upon by an operator \( \vec{S}_i \):
  - full quantum space is \( \mathcal{H} = V_1 \otimes \cdots \otimes V_N = (\mathbb{C}^2)^\otimes N \);
- nearest neighbours interaction corresponds to the Hamiltonian

\[
H_N = \frac{1}{4} \sum_{i=1}^{N} (1 \otimes 1 - \vec{S}_i \cdot \vec{S}_{i+1}).
\]
XXX$_{1/2}$ spin chain: Lax op’s, monodromy, transfer matrix

Lax operators

$$\mathcal{L}_{ai} \in \text{End} \left( V_{\text{aux}} \otimes V_i \right) , \quad \mathcal{L}_{ai} = \left( u - \frac{i}{2} \right) 1 \otimes 1 + i P_{ai}$$

Monodromy matrix

$$\mathcal{M}_a(u) \in \text{End} \left( V_{\text{aux}} \otimes \mathcal{H} \right) , \quad \mathcal{M}_a(u) = \mathcal{L}_{aN}(u) \cdots \mathcal{L}_{a1}(u)$$

Transfer matrix

$$T(u) \in \text{End} \left( \mathcal{H} \right) , \quad T(u) = \text{tr}_a \mathcal{M}_a(u)$$

$$[T(u), T(u')] = 0$$
XXX_{1/2} spin chain: Algebraic Bethe Ansatz

Let |\Omega\rangle = |\uparrow \cdots \uparrow\rangle be a vacuum state.

\[ \mathcal{M}_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \]  

\[ A(u) |\Omega\rangle = \alpha(u) |\Omega\rangle \]
\[ D(u) |\Omega\rangle = \delta(u) |\Omega\rangle \]

\[ B(u_1) \cdots B(u_M) |\Omega\rangle \] is an eigenvector of \( T(u) = A(u) + D(u) \) if and only if \( u_1, \ldots, u_M \) are roots of the Bethe equations

\[ \left( \frac{u_i + i/2}{u_i - i/2} \right)^N = \prod_{k \neq i} \frac{u_i - u_k + i}{u_i - u_k - i} , \quad i = 1, \ldots, M . \]
XXX_{1/2} spin chain: Yangian symmetry

Expansion of the monodromy in the limit $u \to \infty$ yields

$$u^{-N} \mathcal{M}_a(u) = 1 + \frac{1}{u} \mathcal{J}^{(0)} + \frac{1}{u^2} \mathcal{J}^{(1)} + \mathcal{O} \left( \frac{1}{u^3} \right)$$

$Y(\mathfrak{su}(2))$ is not an exact symmetry of the Heisenberg spin chain, due to the periodic BCs. Still, constraints in the bulk.
Amplituhedron: the spin chain setup

Focus on the amplituhedron $\mathcal{A}_{n,k;m}^{\text{tree}}$ associated to $\Omega_{n,k}^{(m)}$.

- Spin chain of length $n + 1$, inhomogeneities $v_1, \ldots, v_n$;
- at each site, we have $\mathfrak{gl}(m + k)$ representations in terms of differential operators, whose generators are

$$ (J_i)^A_B = Z_i^A \frac{\partial}{\partial Z_i^B} , \quad (J_Y)^A_B = \sum_{\alpha} Y_{\alpha}^A \frac{\partial}{\partial Y_{\alpha}^B} + k\delta^A_B ; $$

- the quantum space is

$$ \mathcal{H} = V_1 \otimes \cdots \otimes V_n \otimes V_Y , $$

$V_i, V_Y$ are spaces of functions of bosonized mom. twistors.
Amplituhedron: Lax operators et al.

Lax operators

$$\mathcal{L}_i \in \text{End}(V_{\text{aux}} \otimes V_i) \quad , \quad \mathcal{L}_i(u - v_i)^A_B = (u - v_i)\delta^A_B + Z^A_i \frac{\partial}{\partial Z^B_i}$$

Also, define

$$\left(\mathcal{L}_Y\right)^A_B = (m + k)\delta^A_B + \sum_\alpha Y^A_{\alpha} \frac{\partial}{\partial Y^B_{\alpha}} \quad , \quad \mathcal{B}_{ij}(u) = \left(Z^A_j \frac{\partial}{\partial Z^B_i}\right)^u$$

Intertwining relation

$$\mathcal{L}_i(u - v_i)\mathcal{L}_j(u - v_j)\mathcal{B}_{ij}(v_j - v_i) = \mathcal{B}_{ij}(v_j - v_i)\mathcal{L}_i(u - v_j)\mathcal{L}_j(u - v_i)$$

Monodromy matrix

$$\mathcal{M}(u; v_1, \ldots, v_n) = \mathcal{L}_1(u - v_1) \cdots \mathcal{L}_n(u - v_n)$$
Amplituhedron results (1)

Top-cell ($\sigma_{\text{top}} \in S_n$) representation of the volume function is

$$\Omega_{n,k}^{(m)} = \int \frac{d^{k \times n} c_{\alpha i}}{(1 \cdots k)(2 \cdots k + 1) \cdots (n \cdots k - 1)} \prod_{\alpha = 1}^{k} \delta^{m+k}(Y_A^\alpha - c_{\alpha i}Z_i^A).$$

Starting from the seed ($\sim$ vacuum state $|\Omega\rangle$ in ABA)

$$S_k^{(m)} = \int \frac{d^{k \times k} \beta_{\alpha j}}{(\det \beta)^k} \prod_{\alpha = 1}^{k} \delta^{m+k} \left( Y_A^\alpha - \sum_{j=1}^{k} \beta_{\alpha j} Z_j^A \right).$$

$$\Omega_{n,k}^{(m)}(Y, Z) = \prod_{l=1}^{k(n-k)} B_{i_l j_l}(0) S_k^{(m)}$$

where $(i_l, j_l)$ are the 2-cycles in the decomposition of $\sigma_{\text{top}}$. 
Amplituhedron results (2)

Given the previous definition of monodromy, we derive

\[
\mathcal{M}(u; v_1, \ldots, v_n) \frac{\mathcal{L}_Y \Omega_{n,k}^{(m)}}{(u - v_1) \cdots (u - v_n)} = \mathcal{L}_Y \Omega_{n,k}^{(m)}
\]

The large-\(u\) expansion of the LHS implies

\[
J^{(0)} \mathcal{L}_Y \Omega_{n,k}^{(m)} = 0, \quad J^{(1)} \mathcal{L}_Y \Omega_{n,k}^{(m)} = 0
\]

and we find the generators of the Yangian \(Y(gl(m + k))\)

\[
J^{(0)} = \sum_i Z_i^A \frac{\partial}{\partial Z_i^B},
\]

\[
J^{(1)} = \sum_{i<j} Z_i^A \frac{\partial}{\partial Z_i^C} Z_j^C \frac{\partial}{\partial Z_j^B} + \sum_i v_i Z_i^A \frac{\partial}{\partial Z_i^B}.
\]
Outlook

- At tree-level volume functions are constrained by means of the Capelli differential equations

\[
\det \left( \frac{\partial}{\partial W_{a_\mu}^{A_\nu}} \right)_{1 \leq \nu \leq k+1, 1 \leq \mu \leq k+1} \Omega_{n,k}^{(m)}(Y, Z) = 0, \ W_a^A = (Y_A^\alpha, Z_i^A)
\]

[Ferro, Łukowski, AO, Parisi]

Yangian invariance could help tackling the \( k > 1 \) case.
Outlook

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\]

[Ferro, Łukowski, AO, Parisi]

Yangian invariance could help tackling the \( k > 1 \) case.

- Integrands of scattering amplitudes are Yangian-invariant. What about the corresponding loop-level volume functions?
Thank you!