Long integrations of the Restricted Planar Three Body Problem

Lennon O Naraigh*

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Abstract:

This report treats of several long integrations of the Restricted Planar Three Body Problem (RPTBP) for the case of the particle (the smallest body of the system) in the neighbourhood of the point L4, and for the system at the 4:1 resonance. The integration package mercury6[1] is used. The particle is in the neighbourhood of an equilibrium point, and so the motion is expected to be bounded and quasi-periodic in the sense explained below. This is indeed the case. The output from the integrations is presented in graphical form, demonstrating the bounded, periodic nature of the motion in the neighbourhood of L4.

* Lennon O Naraigh: www.maths.tcd.ie/~tkachev Email: tkachev at maths.tcd.ie

1 mercury6: An integration package by John Chambers, Armagh Observatory. See http://star.arm.ac.uk/~jec/home.html

1. Introduction

The RPTBP concerns three massive bodies. We shall call them the sun, the planet and the particle. The sun is much more massive than the planet, while the particle is of negligible mass compared with the masses of the former two bodies. Thus, the sun and the planet are in circular, planar orbit about their common centre of mass (COM). There are five points in the system where the gravitational forces on the particle cancel: these are the Lagrange points. Three of these lie on the line joining the sun and the planet; the other two are the vertices of equlateral triangles whose common base is the line joining the sun and the planet. For definiteness, we consider one of these points only: the triangular Lagrange point L4.

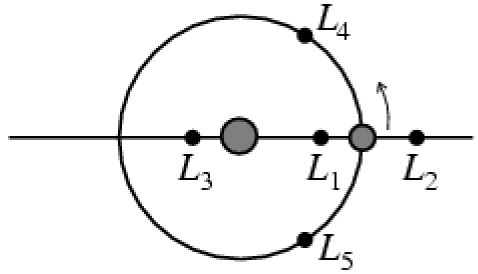

Figure 1. The Lagrange points of the RPTBP. We consider a particle in the neighbourhood of L4.

2. The Equations of Motion in a rotating frame:

We refer to the coordinate systems established by figure 2 below. In non-standard units, the parameters of the equations take the following values:

- Let m_1 denote the mass of the sun and let m_2 denote that of the planet, so that $m_1 > m_2$
- The sun and the planet execute circular orbit of frequency *n* about the common COM.
- We set $n = 1$.
- We choose units such that $\mu = G(m_1 + m_2) = 1$
- Define $\mu_1 = Gm_1$ and $\mu_2 = Gm_2 = \dfrac{m_2}{m_1 + m_2}$.
- Now $m_2 \ll m_1$ in our problem, and so $\mu_1 \approx 1$ and $\mu_2 \ll 1$.

Insert figure 2.

We consider an inertial coordinate system (x^I, y^I, z^I) and a rotating coordinate system (x, y, z) such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^I \\ y^I \\ z^I \end{pmatrix} \tag{1}$$

In this frame and with the units chosen, the equations of motion are as follows:

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2n\dot{y} &= \frac{\partial U}{\partial y} \end{aligned} \tag{2}$$

Where *U* is the *pseudopotential*: $U = -V$, *V* being the potential function.

$$\begin{aligned} U(x, y) &= \frac{1}{2} n^2 (x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \\ r_1 &= \sqrt{(x + \mu_2)^2 + y^2}, \quad r_2 = \sqrt{(x - \mu_1)^2 + y^2} \end{aligned} \tag{3}$$

For the derivation of these equations, see Murray[2].

This system has one conserved quantity, the Jacobi integral, C_J :

$$C_J = -\left(\dot{x}^2 + \dot{y}^2\right) + 2\,U(x, y) \tag{3}$$

There are thus two degrees of freedom (the particle being confined to a plane) and one conserved quantity, so the system of equations in (2) is not integrable.

In the neighbourhood of L4, we may linearlize the equations of motion and look for bounded motion, since L4 is an equlibrium point. If $(x_{0,} y_0)$ are the coordinates of L4 in the rotating frame, we may consider new generalized coordinates $(X, Y) = (x - x_{0,} y - y_0)$. The potential function, to quadratic order, is thus $V(X, Y) = \frac{1}{2}\left(V_{XX} X^2 + 2 V_{XY} XY + V_{YY} Y^2\right)$, omitting constants. Moreover, the coefficients of the new coordinates in this expression are second derivatives:

$$V_{XX} = \left(\frac{\partial^2 V}{\partial X^2}\right)_{L4}, \quad V_{XY} = \left(\frac{\partial^2 V}{\partial X \partial Y}\right)_{L4}, \quad V_{YY} = \left(\frac{\partial^2 V}{\partial Y^2}\right)_{L4} \tag{4}$$

The linearized equations of motion are thus obtained:

$$\begin{aligned} \ddot{X} - 2n\dot{Y} + V_{XX} X + V_{XY} Y = 0 \\ \ddot{Y} + 2n\dot{X} + V_{XY} X + V_{YY} Y = 0 \end{aligned} \tag{5}$$

In matrix / operator form the system (4) reads

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -V_{XX} & -V_{XY} & 0 & 2n \\ -V_{XY} & -V_{YY} & -2n & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix} \tag{6}$$

We solve this system via the substitution $X = X_0 e^{i\sigma t}, \; Y = Y_0 e^{i\sigma t}$. We are thus reduced to solving an albegraic equation:

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2 Murray, C.D. and S.F. Dermott, 1999, *Solar System Dynamics*. Cambridge University Press, 592pp.

$$\sigma^4-\left(4n^2+V_{XX}+V_{YY}\right)\sigma^2+\left(V_{XX}V_{YY}-V_{XY}\right)=0 \tag{7}$$

Substitution of (4) into (7) gives the following equation:

$$\sigma^4-\sigma^2+\frac{27}{4}\mu_2\left(1-\mu_2\right)=0 \tag{8}$$

From the form of the ansatz, it is clear that the necessary and sufficient condition for linear stability is that the roots of (7) be real. This condition reduces to $\mu_2\le\frac{27-\sqrt{621}}{54}$.

We may choose new generalized coordinates (ξ,η) such that relative to these coordinates, the potential V contains no cross term. Thus, we rotate the coordinates (X,Y) by an angle θ :

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix} \tag{9}$$

The angle θ is given by the expression

$$\tan 2\theta=\frac{2V_{XY}}{V_{XX}-V_{YY}} \tag{10}$$

(ξ,η) are the principal axes of the quadratic form $V(X,Y)$.

In SI units[3], the second derivatives have the following form:

$$\begin{aligned}V_{XX}&=-n^2+\frac{1}{4}\frac{GM}{d^3}\\ V_{YY}&=-n^2-\frac{5}{4}\frac{GM}{d^3}\\ V_{XY}&=\frac{3\sqrt{3}}{4}G\frac{\left(m_{small}-m_{big}\right)}{d^3}\end{aligned} \tag{11}$$

Where the symbols have the following meanings:

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3 It is necessary to consider SI units at this point, since the integrator Mercury accepts input data in this standard form.

$m_{small} \equiv m_{2,}\ m_{big} \equiv m_1$, $M \equiv m_{small} + m_{big}$ and where d is the sun-planet distance.

The secular equation (8) takes the following form:

$$\sigma^4 - \frac{GM}{d^3}\sigma^2 + \frac{27}{4}\frac{Gm_{small}}{d^6}\left(GM - Gm_{small}\right) = 0 \qquad (12)$$

If σ_1 and σ_2 are the two linearly independent solutions of this equation, then $\sigma_1^2 + \sigma_2^2 = \frac{GM}{d^3}$ and $\sigma_1^2 \sigma_2^2 = \frac{27}{4}\frac{Gm_{small}}{d^6}\left(GM - Gm_{small}\right)$ (8). These are two equations in two unknowns, and so the frequencies σ_1 and σ_2 can be written in terms of the known parameters of the system. Hence, *X* and *Y* are determined.

3. The 4:1 Resonance

We fix m_{small} and m_{big} so that $\frac{\sigma_1}{\sigma_2} = \frac{1}{4}$. Henceforth, we shall set m_{big} equal to one solar mass and *d* equal to 1AU[4].

Putting $\sigma_1^2 + \sigma_2^2 = a$ and $\sigma_1^2 \sigma_2^2 = b$ we have $\left(\frac{\sigma_2^2}{\sigma_1^2}\right) = \frac{b}{\left(\sigma_1^2\right)^2} = b\left(\frac{17}{a}\right)^2$, and this expression is equal to 16 for the 4:1 resonance. Solving for $r \equiv \frac{m_{small}}{m_{big}}$, we get

$$r = 0.0083393425 \quad .$$

The trajectory of the particle for the 4:1 resonance, with initial conditions close to L4, viewed in the rotating frame, is plotted below.

Inserg figure 3.

4 The natural unit of length in Mercury is 1AU, while the natural unit of mass is the solar mass. This is because Mercury is set up to integrate the solar system.

4. The initial conditions of the integration:

In *Mercury*, the initial conditions of the bodies are specified with respect to the central body (the sun). Now relative to an inertial frame whose origin coincides with the location of the centre of the sun at time zero, L4 has the following coordinates:

$$x = 0.5\,d\,,\quad y = \frac{\sqrt{3}}{2}d\,,\quad z = 0$$

$$u = nd\sin\left(\frac{\pi}{3}\right),\quad v = nd\sin\left(\frac{\pi}{3}\right),\quad w = 0$$

Where $u = \dot{x}$ etc. n is the period of the circular motion of the sun and planet. By Keper's Law, $n = \sqrt{\frac{GM}{d^3}} = \sqrt{\frac{Gm_{big}(1+r)}{d^3}}$.

Relative to this same frame, we choose the following initial conditions for the particle:

$$x=(0.5+\epsilon)d\,,\quad y=\frac{\sqrt{3}}{2}d\,,\quad z=0$$
$$u=nd\sin\left(\frac{\pi}{3}\right),\quad v=nd\sin\left(\frac{\pi}{3}\right),\quad w=0 \tag{8}$$

Where ε is a small perturbation. Happily, these coordinates are those used if we wish to specify the particle with respect to the sun.

We present the numerical values suggested by equations (8) below:

$$x=0.5+\epsilon\,,\quad y=0.8666025403 \quad \text{in AU}$$
$$u=-1.495986742\times10^{-2,}\quad v=8.637083485\times10^{-3} \quad \text{in AU / day}$$

$$m_{big}=0.0083393425 \quad \text{in Solar Masses}$$

5. Integrations:

Mercury is used to integrate the system for a time period of 100,000 (earth) years. The perturbation ε is varied: ε = 0.01, 0.005,0.00025. The motion of the particle fails to be bounded for ε = 0.02. The output files give the coordinates of the particle and the planet, specified w.r.t. the COM. A C program is written to rotate the particle coordinates, to compute the Jacobi integral and to find the principal axes (ξ,η) , The program, together with a readme file, can be obtained at www.maths.tcd.ie/~tkachev/Meteorology.html.

We present some results below, in graphical form, for ε = 0.005. The Jacobi integral is seen to be conserved, while the plots of ξ and η against time exhibit oscillation within an envelope curve. This dramatic envelope is a numerical artifice: the behaviour of the system, as determined by the numerical integration and the initial conditions derived above, is near, but is not at, resonance. This can be seen by examining two elliptic motions, superimposed one on the other: at a resonance, the plots of ξ and η do not have an envelope curve, while close to resonances, this envelope curve appears. Thus, linear theory is adequate in describing the behaviour of the particle near L4 at the 4:1 resonance; at other resonances, however, an analytic examination of the non-linear equations (2) and (3) might be necessary.

Insert fig4&c.

6. Conclusions

The equations of motion of the RPTBP were examined and linearized for motion near L4. These equations were integrated numerically for the case of the 4:1 resonance. The time-variation of the generalized coordinates (ξ, η) exhibts an envelope curve, and this is thought to be a consequence of "de-tuning": the system is near, but not exactly at, resonance. The non-linearity of the equations needs to be attacked in an analytical, rather than numerical way: other resonances may exhibit non-linear effects not considered here.