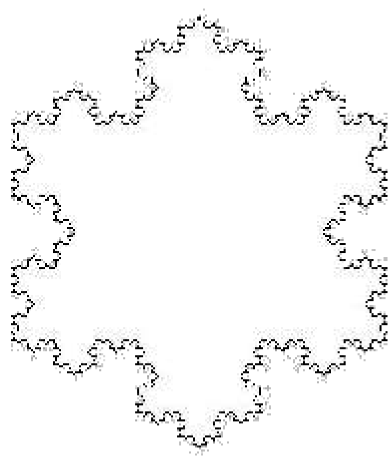
Fractals – Lennon O Naraigh (JSTP)

Abstract:

This experiment seeks to analyze the morphology of zinc electrodeposits using fractal geometry. The zinc electrodeposits are formed and are seen to assume four structures: dense radial, diffusion-limited aggregation (DLA), dendritic and stringy. The structure-formation as a function of voltage and the concentration of the zinc sulphate in an aqueous solution is observed and a phase-diagram of this dependence is plotted below.

Basic theory and equations[1]:

An example of a fractal is the Koch curve, a stage of whose generation is shown below.



Thus the curve encloses a finite area but is of infinite length. Further, the Koch curve is everywhere continuous but nowhere differentiable. More worrying still is the prospect that this curve has a non-integral **dimensionality**.

Dimensionality in Euclidean space:

Consider a line segment of length a_0 in ordinary (Euclidean) space. Divide the line into a large number of equal subdivisions of length $a << a_0$. Similarly, consider a square of sides of length a_0 in space and divide it up into many small squares of sides of length a, with $a << a_0$. Finally, consider a cube with sides of length a_0, and divide it up into many smaller cubes of sides of length a, with $a << a_0$. In each case, the number of subdivisions, written as $N(a)$, is given by

$$N(a) = \left(\frac{a_0}{a} \right)^d \quad (1)$$

d is the dimensionality of the object and for the cases here is either 1, 2 or 3.

Thus, taking (natural) logs on both sides, we get the following:

$$d = \frac{\log N(a)}{\log(a_0 / a)}.$$

If we determine the dimensionality of a given object using the above formula, we call this the **fractal dimension**, d_F. We therefore have the identity

$$d_F \equiv \frac{\log N(a)}{\log(a_0 / a)} \tag{2}$$

We can therefore *define* a **fractal**: it is an object or a set with a non-integral fractal dimension that demonstrates the property of self-similarity.

If a fractal curve has dimensionality d_F in a Euclidean space, we shall call the dimensionality of that space d_E . I.e. d_E is the dimension of the space in which the fractal object is embedded. We have that $d_F < d_E$.

A prime example is the **Cantor set**, which is got by considering the interval [0,1] of the real line. We remove the middle third of this interval. We then remove the middle thirds of the remaining two intervals, and so on. The set is self-similar in the sense that if we consider the set at any stage of its generation, and at some later stage, then a magnification of the set at this later stage will have the same appearance as the set at the previous stage of generation.

We can compute the fractal dimension of the Cantor set: We consider the n^{th} subdivision – of length a – of the line of length 1, and so we have that $a = \frac{1}{3^n}$, and $N(a) = 2^n$. Thus, $d_F = \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3}$, which holds for all values of n.

For the Koch curve, we have, for the n^{th} stage of its generation, that $N(a) = 4^n$. Since $a = \frac{a_0}{3^n}$, we have that $\frac{a_0}{a} = 3^n$, and so $d_F = \frac{\log 4}{\log 3}$

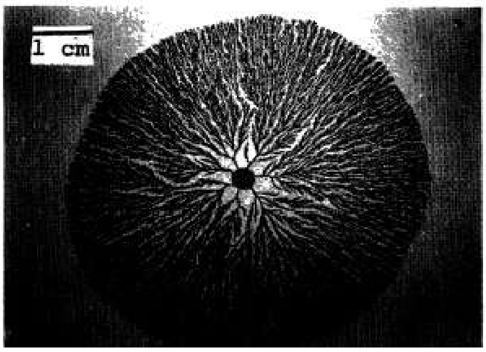
There is a two-dimensional analogue of the Cantor set: we consider the unit square in $\mathbf{R}^2$ and divide it up into $3^2 = 9$ sub-squares. We then remove the middle sub-square and repeat the procedure for the remaining eight squares. This procedure is repeated indefinitely and a fractal dimension $d_F = \frac{\log 8}{\log 3}$ is obtained. This fractal is called the *Sierpinski carpet*. It is easy to generate a Sierpinski carpet with Euclidean dimension N. Its fractal dimension is then $d_F = \frac{\log(3^N - 1)}{\log 3}$ (3).

We can calculate the fractal dimension d_F by the so-called **box-counting method**.

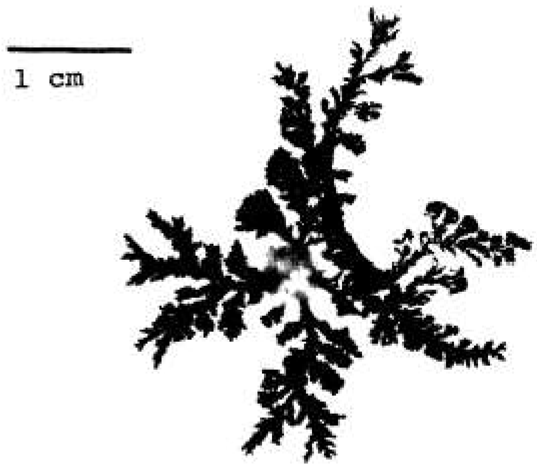
We construct boxes of length l (or alternatively, we can construct spheres the diameter of whose great circles is l) in such a way as to cover the fractal curve. We then find the minimum number of boxes required.

Let l be the length of the box and let N be the number of boxes. We find that the following result holds: $\log N \propto \log \frac{1}{l}$. This is a consequence of equation (2). Plotting $\log N$ against $\log \frac{1}{l}$ gives the slope of the line, D. Thus, if $N(l) \propto l^{-D}$, then, in the limit as $l \to 0$, D is called the **box-counting fractal dimension**.

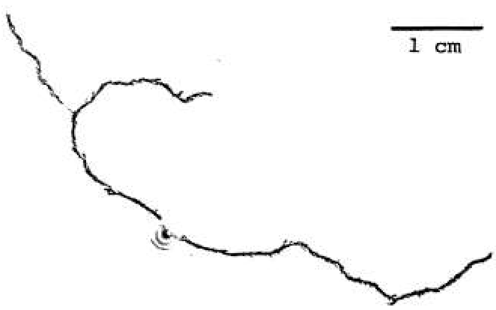
Now this discussion focuses on mathematical objects – ideal fractals. The structure the zinc assumes in this experiment cannot be self-similar on all scales – the finite size of the structure sets an upper limit on the self-similarity. As mentioned, the fractal structure falls roughly into four categories – dense radial, DLA, dendritic and stringy. These expected structures are shown below:



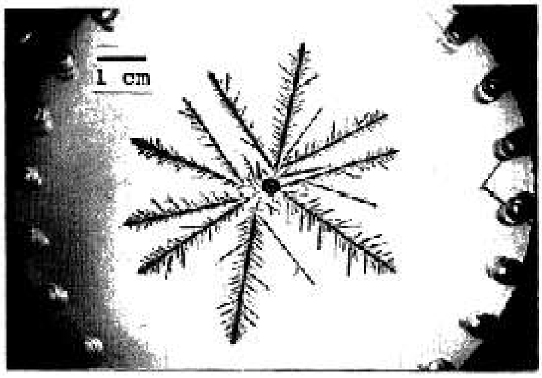
(a) Dense Radial



(b) Diffusion Limited



(c) Stringy



(d) Dendritic

Figure 1: Expected fractal patterns[2].

The DLA structure is modelled as follows: given a sea of particles in random motion, we can imagine one fixed particle at the centre. This particle becomes "sticky" and, when another particle comes in contact with this central particle, the latter particle stops moving and becomes another "sticky" site. Thus, the fractal grows radially outward.

Method:

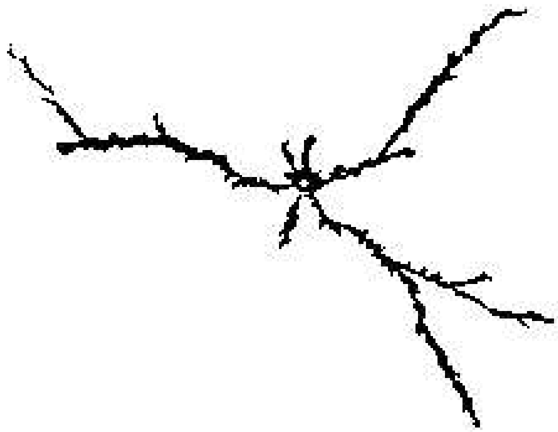
Aqueous solutions of zinc sulphate of varying concentrations are made. The fractal structures are then grown by applying a voltage across the solution. The field is radial. When the deposit is grown, an image is obtained using a CCD camera. The image is saved to a computer and is analyzed using a program which calculates the fractal dimension of the structure.

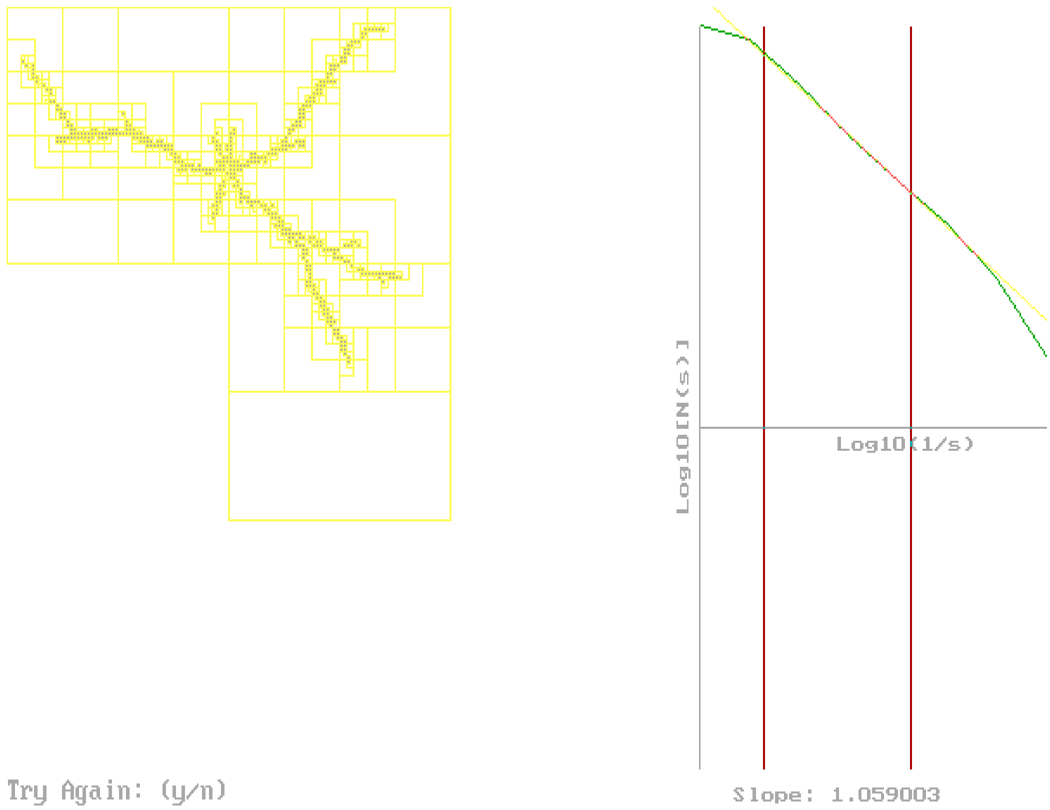
Results:

Here are the images of the fractals obtained for a given voltage and concentration of the zinc sulphate in the aqueous solution. In each case, the slope gives the fractal dimension, d_F.

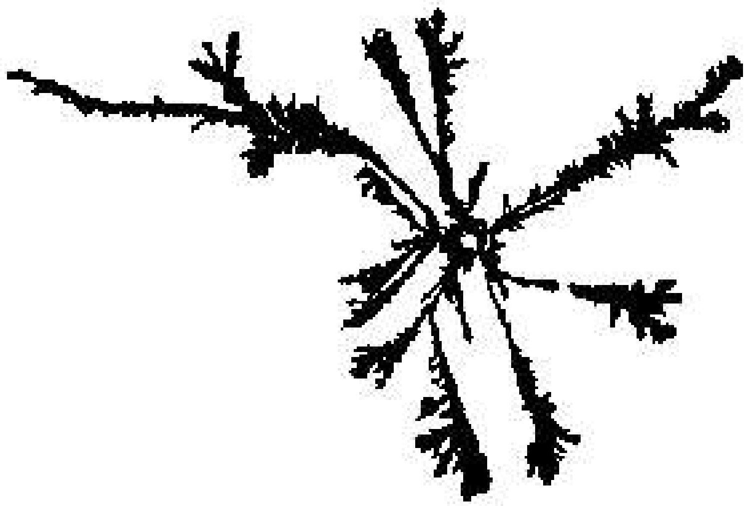
Figure 2:

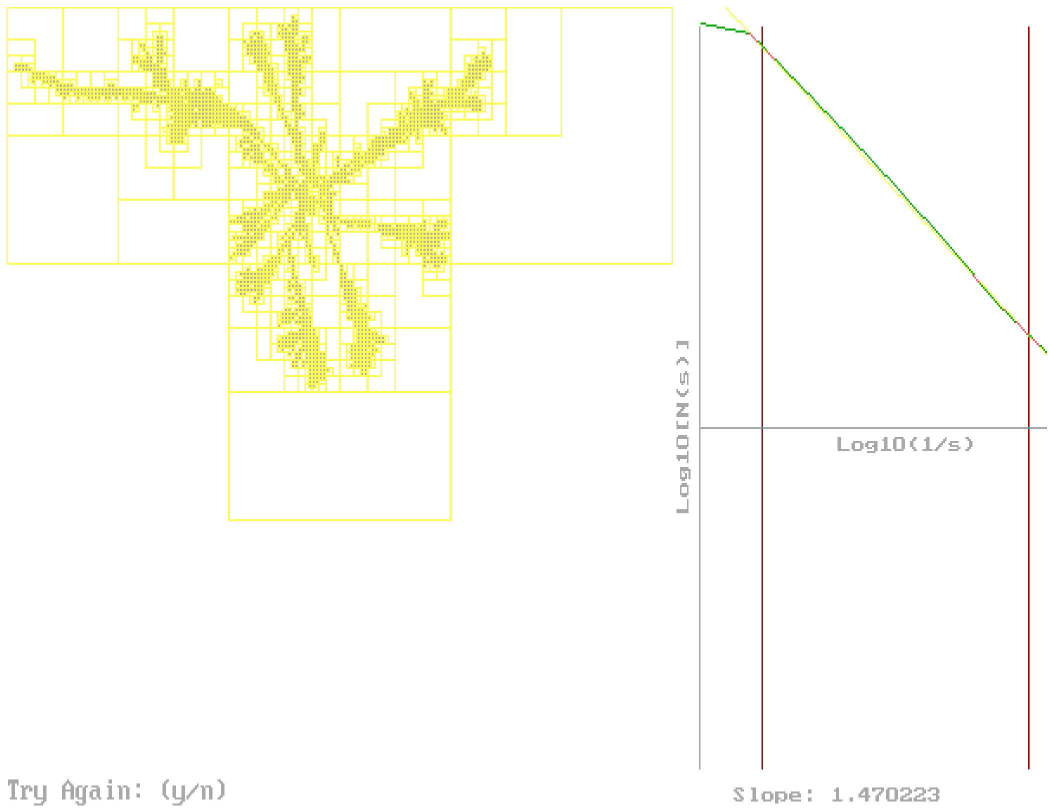
(i) 5V, 1 molar solution. Stringy



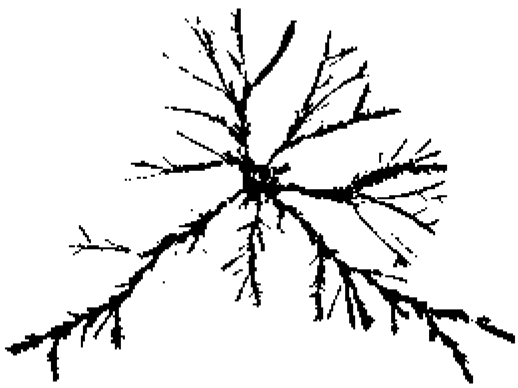


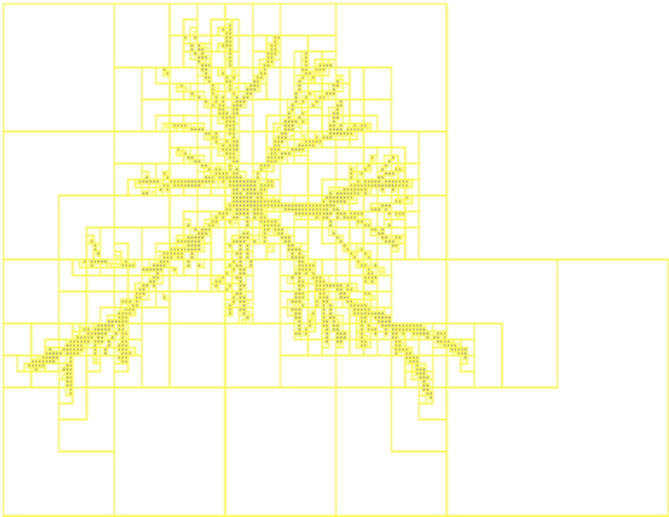
(ii) 2.5 V, 1 molar solution. DLA





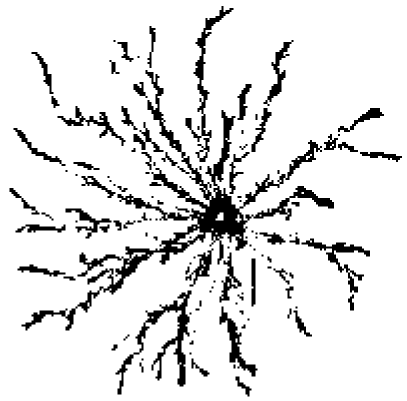
(iii) 5V, 0.5 molar solultion. Dendritic

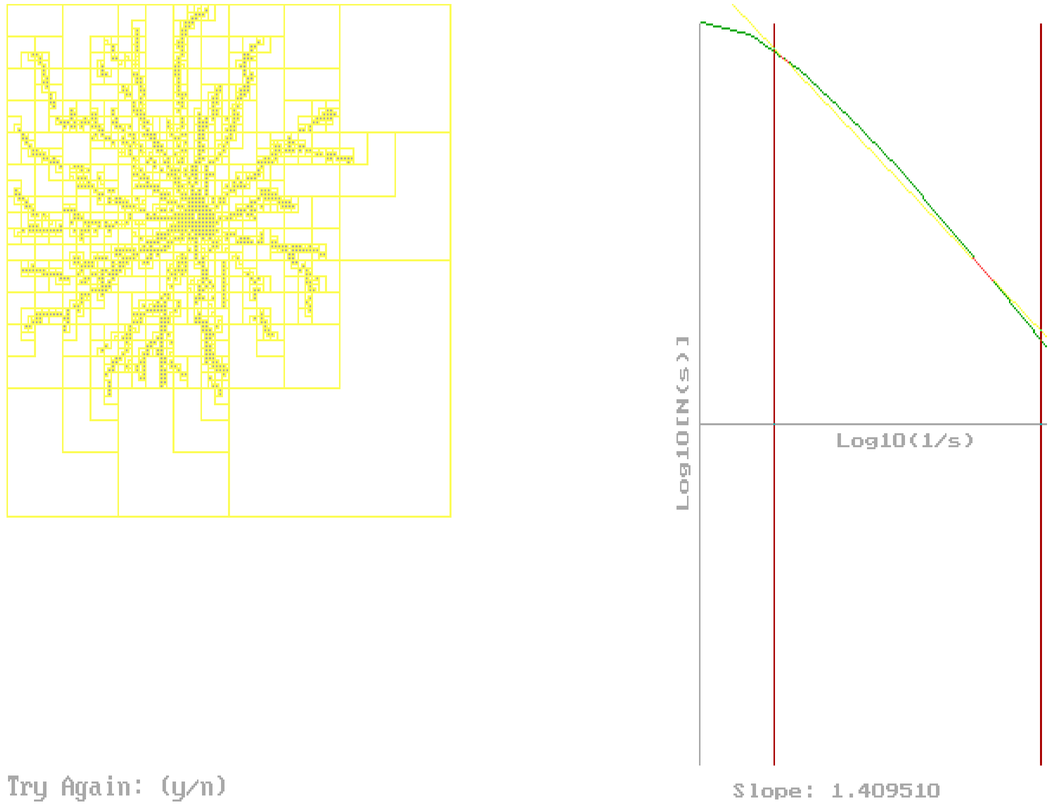




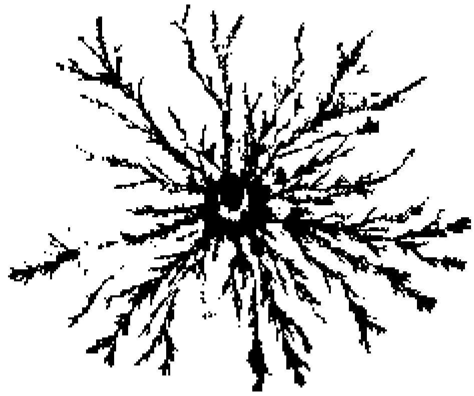
Try Again: (y/n)

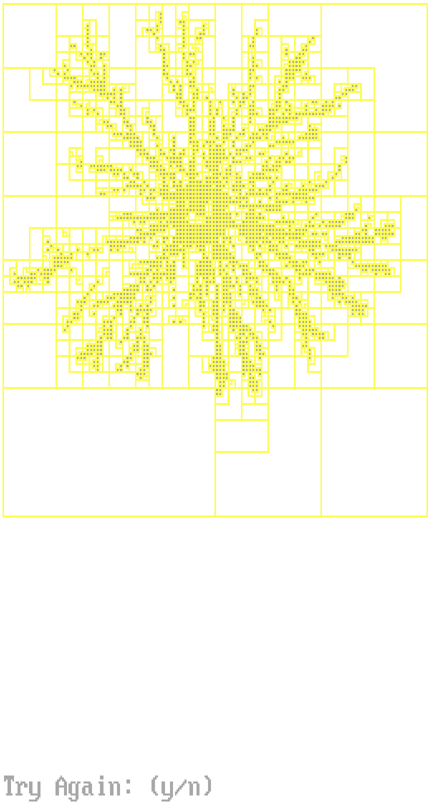
(iv) 15V, 0.5 molar solution. Stringy



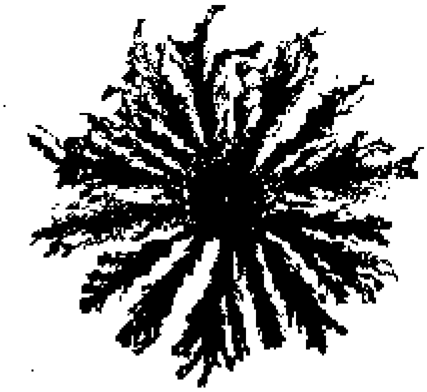


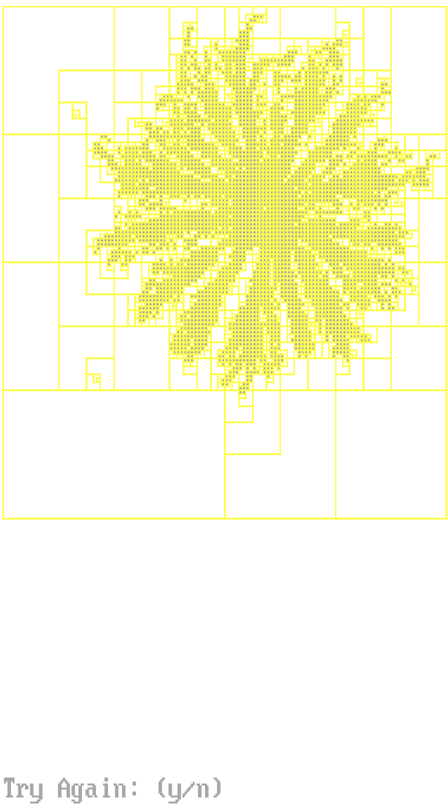
(v) 5V, 0.1 molar solution. Dense radial



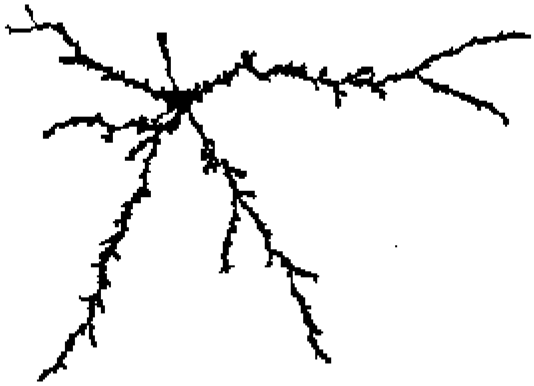


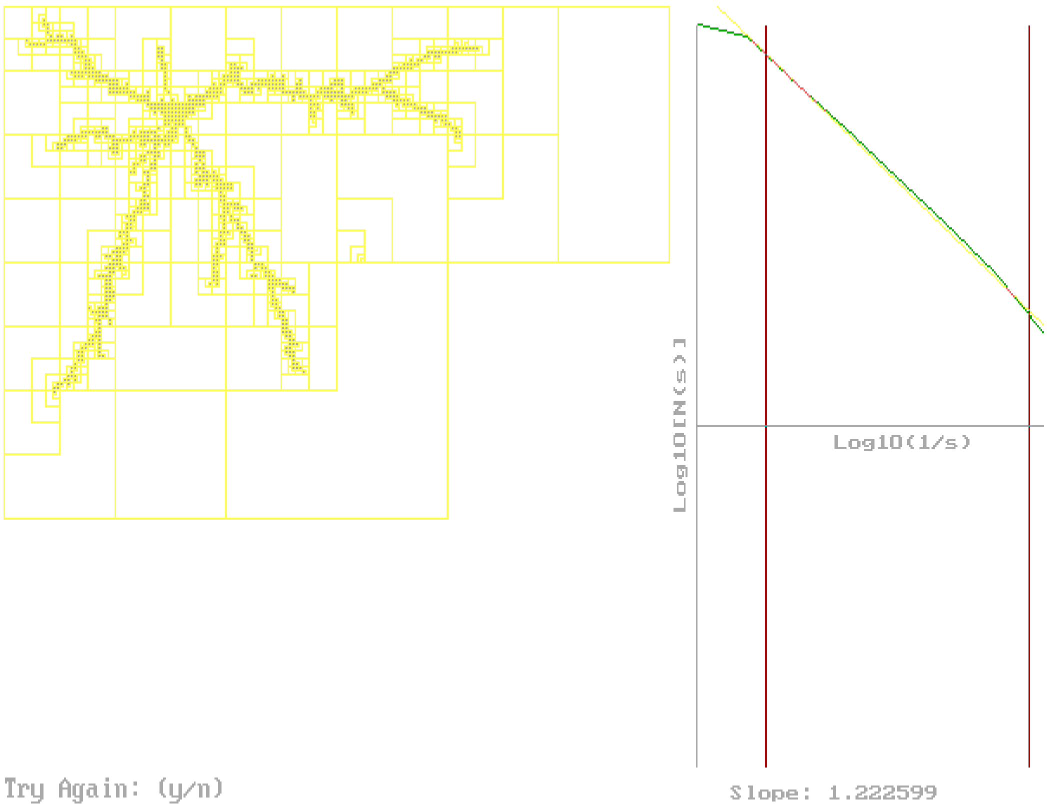
(vi) 15V, 0.1 molar solution. Dendritic.



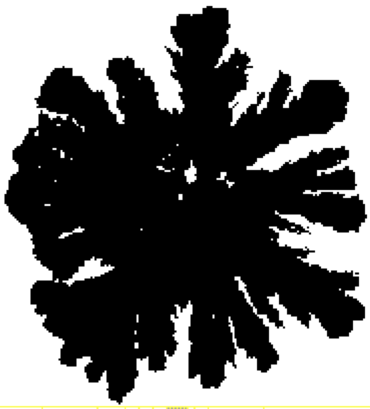


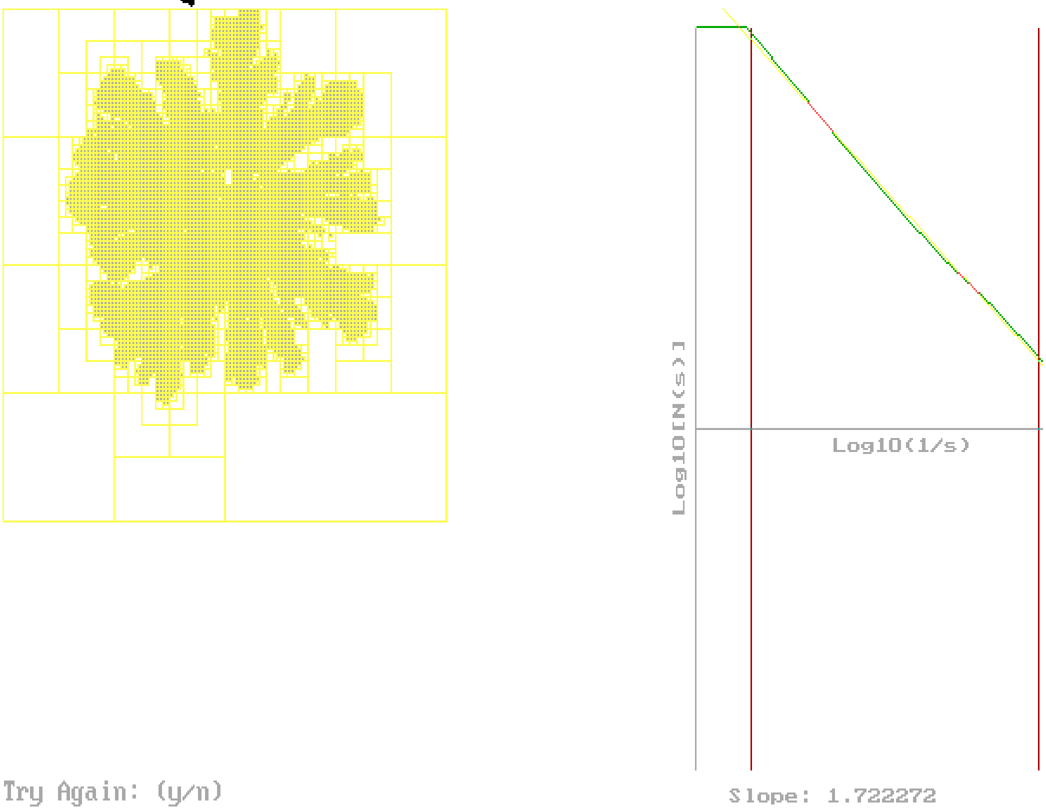
(vii) 11V, 1 molar solution. Stringy



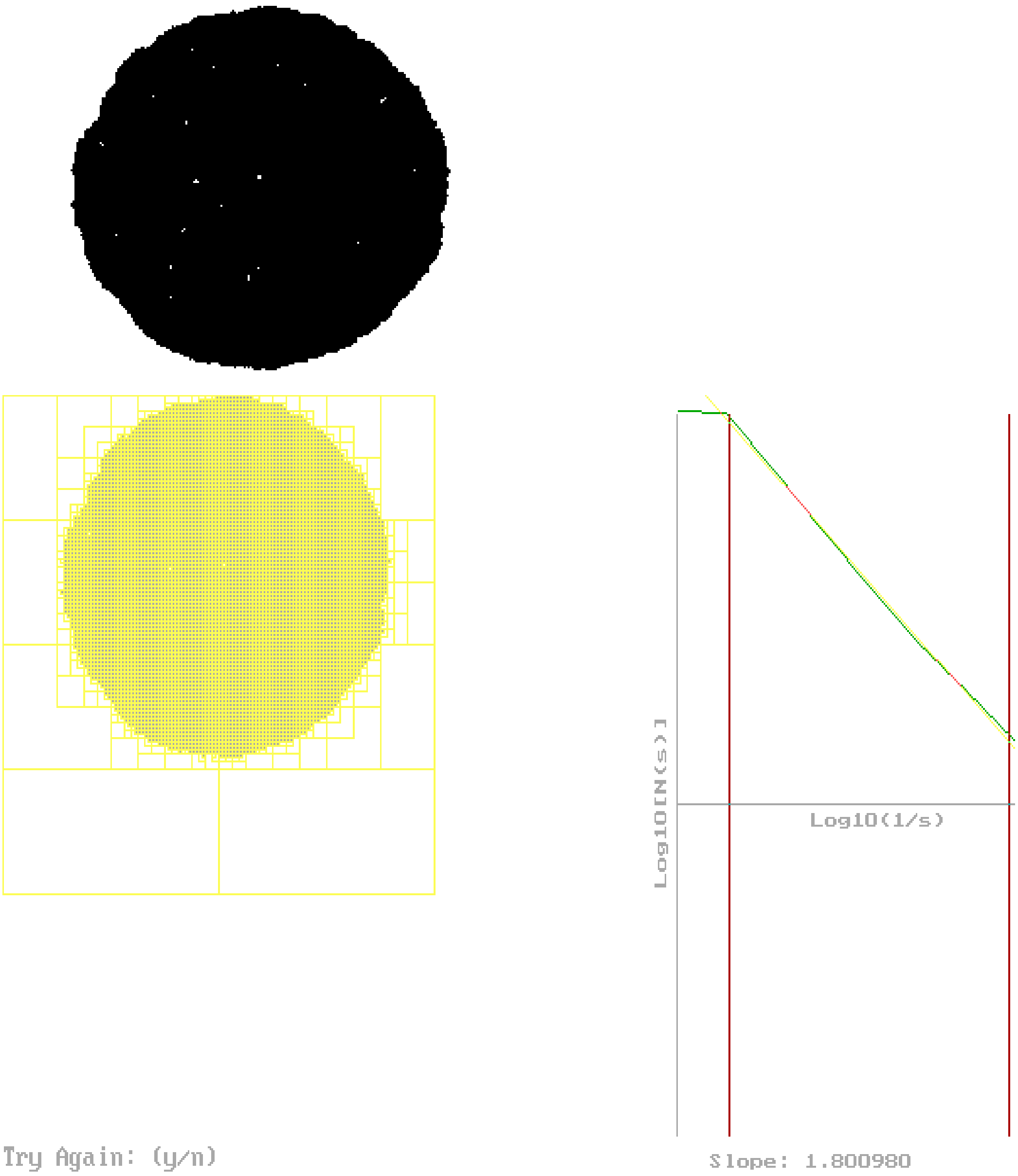


(viii) 5V, 0.05 molar solution. Dense radial.





(ix) 15V, 0.05 molar solultion. Dense radial.



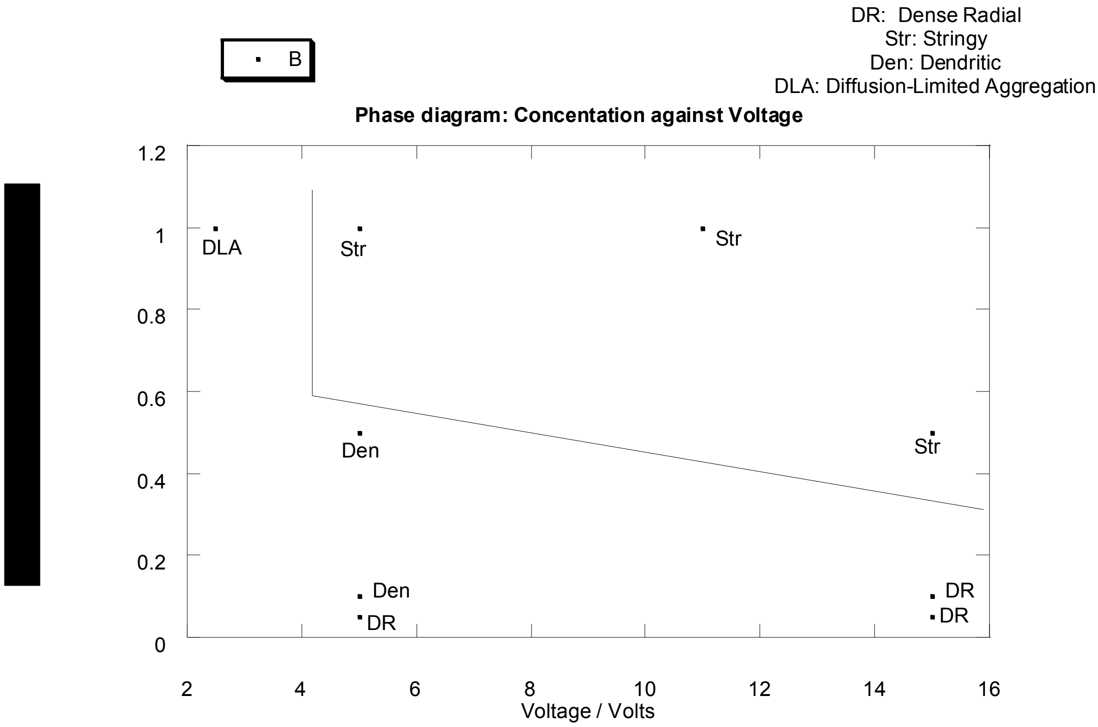
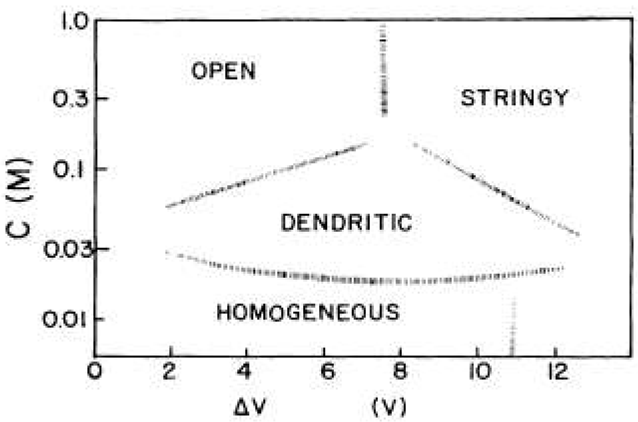


Figure 3: The phase diagram. Too few data points were obtained to divide up the the space conclusively. (For example, D Grier et al.[4]took 250 data points.)However, the results compare well with those shown below[5] (for the same zinc sulphate).



Here, "open" refers to the DLA structure and "homogeneous" refers to the dense radial structure.

Conclusion:

The fractal dimension of each of the grown fractals was found. The phase diagram for the fractal structure as a function both of applied voltage and zinc sulphate concentration was plotted, and was found to agree with the expected plot.

References:

[1] Herbert Goldstein, *Classical Mechanics* (Addison Wesley, 2002), *passim*.
[2] D Grier et al., *Morphology and Microstructure in Electrochemical Deposition of Zinc*, Phys. Rev. Lett., **56**, 1264 – 1267 (1986).
[3] *Ibid*.
[4] Yasuji Sawada et al., *Dendritic and Fractal Patterns in Electrolytic Metal Deposits*, Phys. Rev. Lett., **56**, 1260 – 1263 (1986).