1S1 Tutorial Sheet 9, Solutions[1]

14-16 December 2011

Questions

1. (2) Find the indefinite integral of $x^3 - 4x$, $\sqrt{x} + 1/\sqrt{x}$, $x^3 - \sqrt[3]{x}$ and $x(1+x^4)$.

   Solution: All these functions can be integrated using the power rule and linearity of the indefinite integral

$$\int (x^3 - 4x)\,dx = \frac{1}{4}x^4 - 2x^2 + C \qquad (1)$$

$$\int (\sqrt{x} + 1/\sqrt{x})\,dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C \qquad (2)$$

$$\int (x^3 - \sqrt[3]{x})\,dx = \frac{1}{4}x^4 - \frac{3}{4}x^{4/3} + C \qquad (3)$$

$$\int x(1+x^4)\,dx = \frac{1}{2}x^2 + \frac{1}{6}x^6 + C \qquad (4)$$

2. (2) Evaluate

$$\int \frac{1}{1-\sin x}dx \qquad (5)$$

   Hint: first multiply by $1 + \sin x$ above and below the line, then use the identity $\cos^2 x + \sin^2 x = 1$.

   Solution: We first transform the integrand:

$$\frac{1}{1-\sin x} = \frac{1}{1-\sin x}\frac{1+\sin x}{1+\sin x} = \frac{1+\sin x}{1-\sin^2 x} = \sec^2 x + \tan x \sec x. \qquad (6)$$

   From the integral table we then find

$$\int \frac{1}{1-\sin x}dx = \int (\sec^2 x + \tan x \sec x)\,dx = \tan x + \sec x + C \qquad (7)$$

3. (2) Integrate $\tan^2 x$ and $\cot^2 x$.

   Hint: start by using the identity $\cos^2 x + \sin^2 x = 1$ to rewrite the integrand.

   Solution: We first transform

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \sec^2 x - 1, \qquad (8)$$

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so the antiderivative for $\tan^2 x$ is $\tan x - x + C$. Similarly, for $\cot^2 x$ we have

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{1-\sin^2 x}{\sin^2 x} = \text{cosec}\,^2 x - 1, \tag{9}$$

so the antiderivative for $\cot^2 x$ is $-\cot x - x + C$.

4. (2) Solve the following integrals using u-substitution:

$$\int (x^2-1)^{1/3} x\, dx, \qquad \int \sin^2(x)\cos(x)\, dx \tag{10}$$

Hint: use substitutions $u = x^2 - 1$ and $u = \sin(x)$, respectively

Solution: With $u = x^2 - 1$ we have $du = 2xdx$, so that

$$\int (x^2-1)^{1/3} x\, dx = \frac{1}{2}\int u^{1/3}\, du = \frac{1}{2}\frac{3}{4}u^{4/3} + C = \frac{3}{8}(x^2-1)^{4/3} + C \tag{11}$$

and for the second integral we obtain with $u = \sin(x)$ and $du = \cos(x)\, dx$

$$\int \sin^2(x)\cos(x)\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C \tag{12}$$

Extra Questions

The questions are extra; you don't need to do them in the tutorial class.

1. Integrate x^4, $1/x^4$, $\sqrt[4]{x}$ and $1/\sqrt[4]{x}$.

Solution:

$$\int x^4 dx = \frac{1}{5}x^5 + C \tag{13}$$

$$\int \frac{dx}{x^4} = -\frac{1}{3}x^{-3} + C \tag{14}$$

$$\int \sqrt[4]{x} dx = \frac{4}{5}x^{5/4} + C \tag{15}$$

$$\int \frac{dx}{\sqrt[4]{x}} = \frac{4}{3}x^{3/4} + C \tag{16}$$

2. Find y when

$$\frac{dy}{dx} = \sec^2 x - \sin x \tag{17}$$

with $y(\pi/4) = 1$.

Solution: Again, integrate first

$$y = \tan x + \cos x + C \tag{18}$$

and then substitute the condition at $x = \pi/4$

$$1 = 1 + \frac{1}{\sqrt{2}} + C \tag{19}$$

so $C = -1/\sqrt{2}$ and

$$y = \tan x + \cos x - \frac{1}{\sqrt{2}} \tag{20}$$

3. Use the identities $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ to integrare $\sin^2(x/2)$ and $\cos^2(x/2)$.

   Solution: So

$$\int \sin^2(x/2)dx = \frac{1}{2}\int (1 - \cos x)dx = \frac{1}{2}(x - \sin x) + C \tag{21}$$

   and

$$\int \cos^2(x/2)dx = \frac{1}{2}\int (1 + \cos x)dx = \frac{1}{2}(x + \sin x) + C \tag{22}$$

4. Suppose a point moves along a curve $y = f(x)$ in the xy-plane in such a way that at each point (x, y) on the curve the tangent line has slope $-\sin x$. Find an equation for the curve, given that it passes though the point $(0, 2)$.

   The description of the curve tells us that $y' = -\sin x$, so $y = \cos x + C$ and $y(0) = 2 = 1 + C$ so $C = 1$ and

$$y = \cos x + 1 \tag{23}$$