MA1S11 Tutorial Sheet 8, Solutions[1]

7-9 December 2011

Questions

1. (4) Consider the function

$$f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 + 1 \tag{1}$$

- Determine the relative and the absolute extrema of f on the closed interval $[-2, 3]$.
- Determine the inflection points and the regions of concavity up or down.

Solution:

The function is differentiable everywhere on the open interval $(-2, 3)$. Calculating its derivatives we get

$$f'(x) = -x^3 + x^2 + 2x = -x(x-2)(x+1) \tag{2}$$

$$f''(x) = -3x^2 + 2x + 2 = -3\left(x - \frac{1}{3} - \frac{\sqrt{7}}{3}\right)\left(x - \frac{1}{3} + \frac{\sqrt{7}}{3}\right) \tag{3}$$

Solving $f'(x) = 0$ we get $x = 0, 2, -1$ as possible points. Applying the second derivative test we see that -1 and 2 are relative maxima and 0 is a relative minimum. The values of f at the relative extrema are

$$f(0) = 1, \qquad f(2) = \frac{11}{3}, \qquad f(-1) = \frac{17}{12} \tag{4}$$

This is to be compared with the values of f at the endpoints $-2, 3$:

$$f(-2) = -\frac{5}{3}, \qquad f(3) = -\frac{5}{4}. \tag{5}$$

We conclude that f has an absolute maximum at $x = 2.$ and an absolute minimum at the left endpoint $x = -2$.

To get the inflection points we solve $f''(x) = 0$ and get

$$x = \frac{1}{3} + \frac{\sqrt{7}}{3}, \frac{1}{3} - \frac{\sqrt{7}}{3}, \tag{6}$$

Solving the inequality

$$f''(x) > 0 \quad \Rightarrow \quad \frac{1}{3} - \frac{\sqrt{7}}{3} \approx -0.5485 < x < \frac{1}{3} + \frac{\sqrt{7}}{3} \approx 1.2153 \tag{7}$$

we obtain the region of concavity up, and f is concave down for all other x. The graph of f is given in figure 1

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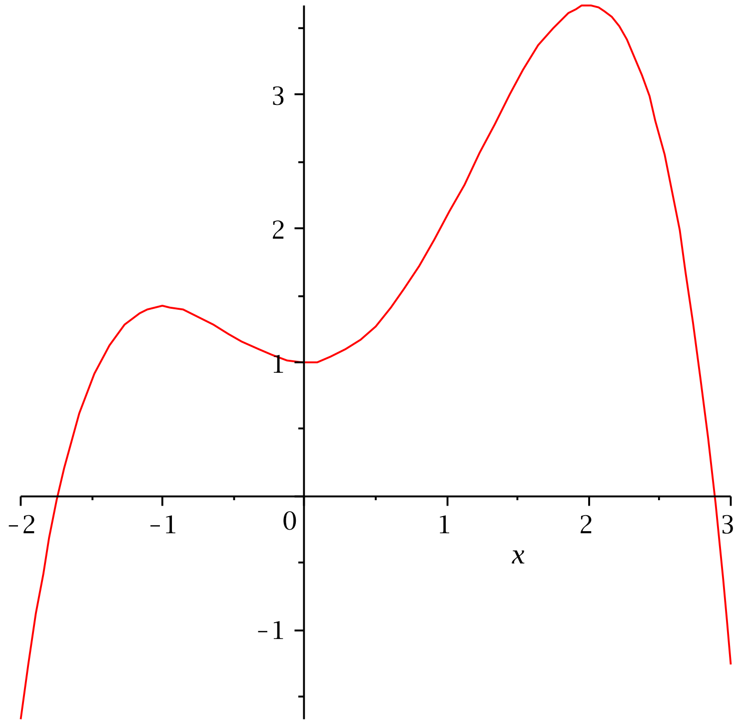

Figure 1: question 1: graph of $f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 + 1$.

2. (4) A person jumps from a height of $10m$ into a swimming pool. The position (in metres above water) as a function of time (in seconds) is given by

$$s(t) = 10m - \frac{1}{2}gt^2, \tag{8}$$

where $g = 9.81m/s^2$ (metres per seconds squared) is the acceleration on earth due to the gravitational force.

Calculate

- How long does the person fall?
- What is the average velocity of the fall (total length divided by duration of the fall)?
- What is the velocity as a function of t (you need to calculate the derivative of $s(t)$ with respect to time)?
- What is the velocity at the end of the fall? How much is this in kilometres per hour?

Solution:

To find out how long the fall is one needs to solve $s(t_0) = 0$, as the person is hitting the water at the corresponding time t_0. Solving

$$s(t_0) = 0 \Leftrightarrow t_0 = \sqrt{20m/g} \approx 1.428s. \tag{9}$$

Therefore the average velocity is $v_{\text{average}} = (10/1.428)m/s \approx 7m/s$. To compute the velocity as a function of time we compute

$$v(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} = -gt, \tag{10}$$

(the minus sign is due to our choice of counting distances from the surface of the water upward, while the movement is downward) so that the velocity at arrival is given by $-v(t_0) = gt_0 = 14m/s$. This corresponds to about 50 kilometres per hour.

Extra Question
Solution:

1. Use Newton's Method to find the root of $x^3 - x + 2$ to three decimal places using -1 as an initial guess.

   Solution:

   We set $f(x) = x^3 - x + 2$ and obtain

$$x_{n+1} = \frac{2(x_n^3 - 1)}{3x_n^2 - 1}. \tag{11}$$

   Starting with $x_1 = -1$ we get

$$\begin{aligned} x_2 &= -2 && (12)\\ x_3 &= -18/11 \approx -1.6363636 && (13)\\ x_4 &= -14326/9361 \approx -1.530392052 && (14)\\ x_5 &\approx -1.521441465 && (15)\\ x_6 &\approx -1.521379710, && (16) \end{aligned}$$

   so that 4-5 steps are enough to reach the required precision. Using Maple or Mat hematica the correct root is given by

$$-\frac{(27 + 3\sqrt{78})^{2/3} + 3}{3(27 + 3\sqrt{78})^{1/3}} \approx -1.521379707, \tag{17}$$

   which shows that x_6 only differs in the 8th digit.