MA1S11 Tutorial Sheet 7, Solutions[1]

30 November - 1 December 2011

Questions

The numbers in brackets give the numbers of marks available for the question.

1. (4) Determine the derivative of the inverse function f^{-1}, by using the chain rule and implicit differentiation of the equation

$$f\left(f^{-1}(x)\right) = x \tag{1}$$

Then use the result to obtain the derivative of the inverse functions of $f(x) = \sqrt{x}$ ($x \geq 0$) and $f(x) = \cos x$ (for $x \in [0, \pi]$).

(Hint: for the inverse cosine function you'll need the identity $\cos^2 x + \sin^2 x = 1$)

Solution: We differentiate both sides of the equation w.r.t. x to obtain

$$f'\left(f^{-1}(x)\right) \frac{df^{-1}(x)}{dx} = 1 \qquad \Leftrightarrow \qquad \frac{d}{dx} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)} \tag{2}$$

In the case $f(x) = \sqrt{x}$ one has $f'(x) = 1/(2\sqrt{x})$ and

$$\frac{d}{dx} f^{-1}(x) = 2\sqrt{f^{-1}(x)} = 2\sqrt{x^2} = 2x \tag{3}$$

For $f(x) = \cos x$ we set $u = f^{-1}(x)$ or $\cos u = x$ and have

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(u)} = \frac{1}{-\sin u} = \frac{1}{-\sqrt{1 - \cos^2 u}} = \frac{-1}{\sqrt{1 - x^2}}, \tag{4}$$

so that we conclude

$$\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}. \tag{5}$$

2. (4) Determine the zeros, the relative maxima and minima, the regions of concavity up/down and the limit behaviour $x \to \pm\infty$ for

$$f(x) = (x + 1)^2(x - 1) = x^3 + x^2 - x - 1 \tag{6}$$

Then draw the graph of f using all the gathered information.

Solution:

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Using the factorised form we read of the zeros of f as $x = 1, -1$. The function is polynomial in x and therefore differentiable everywhere. Critical points are obtained by solving $f'(x) = 0$,

$$f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1) \qquad \Rightarrow \qquad x = 1/3, -1. \tag{7}$$

To find out about the regions of concavity we differentiate again and obtain

$$f''(x) = 6x + 2. \tag{8}$$

Thus $f''(x)$ is positive for $x > -1/3$ and negative for $x < -1/3$; $x = -1/3$ is the point of inflection dividing the regions concave down ($x < -1/3$) and concave up ($x > -1/3$). To determine which of the stationary points are maxima, minima or neither we use the second derivative test to find $f''(1/3) = 4 > 0$ and $f''(-1) = -4 < 0$, so that a relative maximum is obtained at $x = -1$ and a relative minimum at $x = 1/3$. Finally the end behaviour is obtained as

$$\lim_{x\to\infty} f(x) = +\infty, \qquad \lim_{x\to-\infty} f(x) = -\infty \tag{9}$$

as the leading power is x^3. Hence we obtain the following graph:

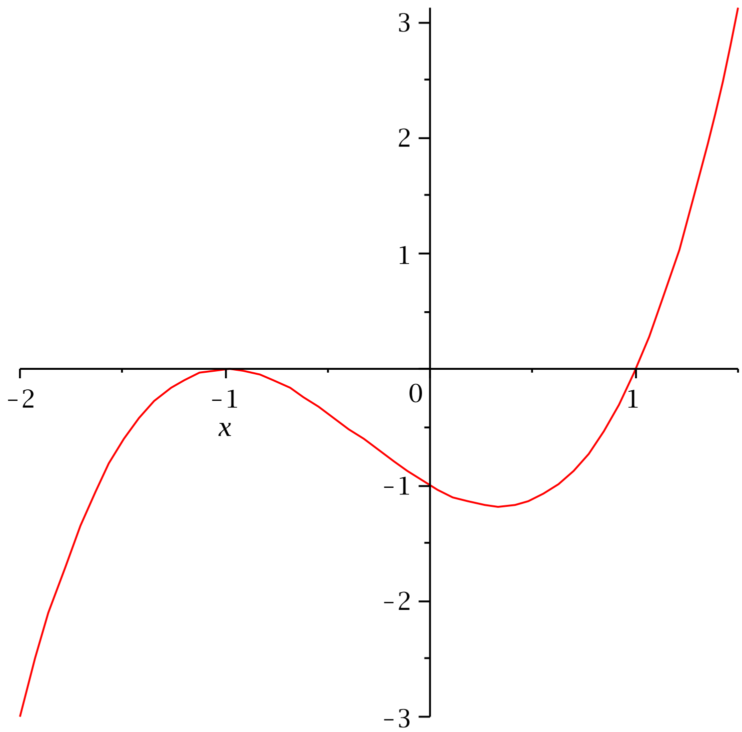

Figure 1: graph of $f(x)$ in question 2.