MA1S11 Tutorial Sheet 6, Solutions[1]

23-25 November 2011

Questions

The numbers in brackets give the numbers of marks available for the question.

1. (4) Differentiate the following functions with respect to x:

$$x^3 \cos 3x, \qquad \sin(\cos x), \qquad \frac{\sin x}{\cos^2 x}, \qquad -x^2 \cos\sqrt{x} \tag{1}$$

Solution: So, differentiating x^3 gives $3x^2$, and differentiating $\cos 3x$ gives $-3\sin 3x$ using the chain rule. Then, by the product rule,

$$\frac{d}{dx}\left[x^3 \cos(3x)\right] = 3x^2\{\cos(3x) - x\sin(3x)\}. \tag{2}$$

Again use the chain rule with $f(x) = \sin x$ and $g(x) = \cos x$,

$$\frac{d}{dx}\sin(\cos x) = -\cos(\cos x)\sin x. \tag{3}$$

In the next one we have an example of the quotient rule with $f(x) = \sin(x)$ and $g(x) = \cos^2 x$, and we need the chain rule to differentiate g,

$$g'(x) = \frac{d}{dx}\cos^2 x = -2\cos x \sin x. \tag{4}$$

Taken together this gives

$$\frac{d}{dx}\frac{\sin x}{\cos^2 x} = \frac{\cos^2 x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos^2 x}{\cos^4 x} = \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} \tag{5}$$

For the last one we need again product and chain rules with $f(x) = -x^2$ and $g(x) = \cos\sqrt{x}$ we have

$$\frac{d}{dx}(-x^2\cos\sqrt{x}) = -2x\cos\sqrt{x} - x^2\frac{d}{dx}\cos\sqrt{x}. \tag{6}$$

The chain rule implies

$$\frac{d}{dx}\cos\sqrt{x} = -\sin(\sqrt{x})\frac{1}{2\sqrt{x}}, \tag{7}$$

so that

$$\frac{d}{dx}(-x^2)\cos\sqrt{x} = -2x\cos\sqrt{x} + \frac{1}{2}x^{3/2}\sin(\sqrt{x}). \tag{8}$$

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2. (2) Find dy/dx for

$$3yx^6 - 4x^2 + 6\sin y^4 = 0 \tag{9}$$

Solution:

Differentiate across so

$$\frac{d}{dx}\left(3yx^6 - 4x^2 + 6\sin y^4\right) = 0 \tag{10}$$

giving

$$3\frac{d}{dx}yx^6 - 4\frac{d}{dx}x^2 + 6\frac{d}{dx}\sin y^4 = 0 \tag{11}$$

Using the product rule

$$\frac{d}{dx}yx^6 = 6x^5y + x^6\frac{dy}{dx} \tag{12}$$

and the chain rule

$$\frac{d\sin y^4}{dx} = 4y^3\cos(y^4)\frac{dy}{dx} \tag{13}$$

and hence

$$18x^5y + 3x^6y' - 8x + 24y^3y'\cos y^4 = 0 \tag{14}$$

giving

$$y' = \frac{8x - 18x^5y}{3x^6 + 24y^3\cos y^4} \tag{15}$$

3. (2) Find the slope, that is dy/dx, of the curve

$$3y^2 - 2x^2 = xy \tag{16}$$

at the point $(1,1)$.

Solution:

So, again, differentiating both sides gives

$$6yy' - 4x = y + xy' \tag{17}$$

or

$$y' = \frac{4x + y}{6y - x} \tag{18}$$

and so at $(1,1)$, substituting $x = 1$ and $y = 1$ we have slope equal to 1.

Extra Questions

The questions are extra; you don't need to do them in the tutorial class.

1. Differentiate
$$\frac{1}{\sqrt{1+x}}, \quad \sin(\cos x), \quad \sin\frac{1}{1+x^2}, \quad \sqrt{\sin x}, \quad \tan 2x = \frac{\sin(2x)}{\cos(2x)}. \tag{19}$$

Solution: The first is again an application of the chain rule, with $f(x) = 1/\sqrt{x}$ and $g(x) = x + 1$. Since $g'(x) = 1$ and
$$f'(x) = \frac{d}{dx}x^{-1/2} = -\frac{1}{2}x^{-3/2}, \tag{20}$$
one has,
$$\frac{d}{dx}\frac{1}{\sqrt{1+x}} = f'(g(x))g'(x) = f'(g(x)) = -\frac{1}{2}(x+1)^{-3/2}. \tag{21}$$
For the next one we use again the chain rule
$$\frac{d}{dx}\sin(\cos x) = \cos(\cos x)(-\sin x). \tag{22}$$
In the following example we need to apply the chain rule and then the quotient and chain rule again
$$\frac{d}{dx}\sin\frac{1}{1+x^2} = \left(\cos\frac{1}{1+x^2}\right)\frac{d}{dx}\frac{1}{1+x^2} \tag{23}$$
and then the quotient for the last term with $f(x) = 1$ and $g(x) = 1 + x^2$:
$$\frac{d}{dx}\frac{1}{1+x^2} = \frac{-2x}{(1+x^2)^2} \tag{24}$$
so that the final result is
$$\frac{d}{dx}\sin\frac{1}{1+x^2} = \left(\cos\frac{1}{1+x^2}\right)\frac{-2x}{(1+x^2)^2} \tag{25}$$
The next example needs again the chain rule with $f(u) = \sqrt{u}$ and $g(x) = \sin(x)$
$$\frac{d}{dx}\sqrt{\sin x} = \frac{\cos x}{2\sqrt{\sin x}}. \tag{26}$$
To differentiate $\tan 2x$ we need the quotient rule with $f(x) = \sin(2x)$ and $g(x) = \cos(2x)$ and the chain rule for the derivatives f' and g':
$$f'(x) = 2\cos(2x), \qquad g'(x) = -2\sin(2x). \tag{27}$$
Then the quotient rule gives
$$\frac{d}{dx}\frac{\sin(2x)}{\cos(2x)} = \frac{2\cos(2x)\cos(2x) + 2\sin(2x)\sin(2x)}{\cos^2(2x)} = \frac{2}{\cos^2(2x)} \tag{28}$$
where I have used $\cos^2(2x) + \sin^2(2x) = 1$ in the last step.

2. Find dy/dx and d^2y/dx^2 for

$$\sin xy = 0 \tag{29}$$

Solution: Differentiating we get

$$y \cos xy + x\frac{dy}{dx}\cos xy = 0 \tag{30}$$

so that

$$y + x\frac{dy}{dx} = 0 \tag{31}$$

and

$$\frac{dy}{dx} = -\frac{y}{x} \tag{32}$$

Now, differentiating equation (31) again gives

$$\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 0 \tag{33}$$

so

$$\frac{d^2y}{dx^2} = -\frac{2}{x}\frac{dy}{dx} = \frac{2y}{x^2} \tag{34}$$