MA1S11 Tutorial Sheet 5, Solutions[1]

16-18 November 2011

Questions

The numbers in brackets give the numbers of marks available for the question.

1. (2) Work out f', f'' and keep going differentiating until you get zero.

$$f(x) = \frac{1}{8}x^8 + x^5 + 4x^2 - x + 4 \tag{1}$$

Solution: We have

$$\begin{aligned} f'(x) &= x^7 + 5x^4 + 8x - 1 \\ f''(x) &= 7x^6 + 20x^3 + 8 \\ f'''(x) &= 42x^5 + 60x^2 \\ f^{(4)} &= 210x^4 + 120x \\ f^{(5)} &= 840x^3 + 120 \\ f^{(6)} &= 2520x^2 \\ f^{(7)} &= 5040x \\ f^{(8)} &= 5040 \\ f^{(9)} &= 0 \end{aligned} \tag{2}$$

All derivatives higher than 9 will also vanish.

2. (2) Differentiate

$$f(x) = \frac{1}{x^2}, \tag{3}$$

and

$$f(x) = x + \frac{1}{x}. \tag{4}$$

Solution:

Writing the function with negative exponents and applying the power rule we get:

$$f(x) = x^{-2} \quad \Rightarrow \quad f'(x) = -2x^{-3}. \tag{5}$$

and

$$f(x) = x + x^{-1} \quad \Rightarrow \quad f'(x) = 1 - x^{-2}. \tag{6}$$

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3. (2) Prove the product rule $(fg)' = f'g + fg'$. (Hint: use the limit definition of the derivative for the product function $f(x)g(x)$ and add and subtract appropriate terms in the numerator such that you get $f'(x)g(x) + f(x)g'(x)$ in the limit $h \to 0$.)

   The derivative is defined by

$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right) g(x+h) + f(x)\left(g(x+h) - g(x)\right)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned} \tag{7}$$

4. (2) $f(x) = \sin^2 x + \cos^2 x$, we know by the Pythagorean theorem that $f(x) = 1$, but check that $f'(x) = 0$ from the original formula with sines and cosines. If $f'(x) = 0$ then $f(x)$ is a constant, check the value of the constant by working out $f(0)$. Is the answer consistent with the Pythagorean theorem?

   Solution:

   Using the product rule for $\sin^2$ and $\cos^2$, we find

$$(\sin^2(x))' = 2\sin'(x)\sin(x) = 2\cos(x)\sin(x) \tag{8}$$
$$(\cos^2(x))' = 2\cos'(x)\cos(x) = -2\sin(x)\cos(x) \tag{9}$$

   hence $f'(x) = (\sin^2(x))' + (\cos^2(x))' = 0$ indeed, so $f(x)$ must be a constant function. Evaluate $f(0) = \sin^2(0) + \cos^2(0) = \cos^2(0) = 1$. So f must be the constant function $f(x) = 1$, in agreement with the Pythagorean theorem.

Extra Question

The question is extra; you don't need to do it in the tutorial class.

1. Iterate the product rule by working out the higher derivatives of a product, $(fg)''$, $(fg)'''$ and $(fg)^{(4)}$. Then use the notation

$$f = f^{(0)}, \quad f' = f^{(1)}, \qquad f'' = f^{(2)}, \ldots \tag{10}$$

and analogously for g. Can you discover a relation to the binomial formula (see "useful facts" section) and thus give an expression for the nth derivative $(fg)^{(n)}$?

Solution:

Iterative application of the product rule gives:

$$\begin{aligned}
(fg)' &= f'g + fg' \\
(fg)'' &= f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg'' = f^{(2)}g^{(0)} + 2f^{(1)}g^{(1)} + f^{(0)}g^{(2)} \\
(fg)''' &= f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg''' \\
&= f^{(3)}g^{(0)} + 3f^{(2)}g^{(1)} + 3f^{(1)}g^{(2)} + f^{(0)}g^{(3)} \\
(fg)^{(4)} &= f^{(4)}g + f'''g' + 3f'''g' + 3f''g'' + 3f''g'' + 3f'g''' + f'g''' + fg^{(4)} \\
&= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \\
&= f^{(4)}g^{(0)} + 4f^{(3)}g^{(1)} + 6f^{(2)}g^{(2)} + 4f^{(1)}g^{(3)} + f^{(0)}g^{(4)}
\end{aligned} \tag{11}$$

This does indeed look very similar to the binomial formula and for general positive integer n one has

$$(fg)^{(n)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{(n-k)}g^{(k)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{(k)}g^{(n-k)}. \tag{12}$$