MA1S11 Calculus, Tutorial Sheet 4, Solutions[1]

2-4 November 2011

Questions
The numbers in brackets give the numbers of marks available for the question.

1. (3) Evaluate the limits

$$\lim_{x\to\infty}\frac{5x^2+1}{2x^2+x-4},\quad \lim_{x\to\infty}\frac{4x^3+x^2}{x^3-x+1},\quad \lim_{x\to\infty}\frac{-3x^5+2x^3-2}{4x^5+x^3+4}. \tag{1}$$

Solution: : The highest powers of x are the same in numerator and denominator of all ratios. Dividing in the first ratio by x^2 we get

$$\lim_{x\to\infty}\frac{5x^2+1}{2x^2+x-4}=\lim_{x\to\infty}\frac{5+1/x^2}{2+1/x-4/x^2}=\frac{5}{2}$$

In the second ratio we divide by x^3:

$$\lim_{x\to\infty}\frac{4x^3+x^2}{x^3-x+1}=\lim_{x\to\infty}\frac{4+1/x}{1-1/x^2+1/x^3}=4$$

and in the third ratio we divide by x^5:

$$\lim_{x\to\infty}\frac{-3x^5+2x^3-2}{4x^5+x^3+4}=\lim_{x\to\infty}\frac{-3+2/x^2-2/x^5}{4+1/x^2+4/x^5}=-\frac{3}{4}$$

2. (3) Given $\epsilon>0$ and the limit L of the function $f(x)$ for $x\to\infty$, find $N>0$ such that $|f(x)-L|<\epsilon$ if $x>N$:

   (a)
$$\lim_{x\to\infty}\frac{1}{(x+2)^2}=0,\quad \epsilon=0.01 \tag{2}$$

   Solution: : We need to work out what the condition $|f(x)-L|<\epsilon$ implies for x, for the given ϵ:
$$\left|\frac{1}{(x+2)^2}\right|<\epsilon \Leftrightarrow (x+2)^2>1/\epsilon$$
   which further simplifies to $x>1/\sqrt{\epsilon}-2$. With $\epsilon=0.01$ this evaluates to $x>8$, so for this ϵ, $N=8$ will do.

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(b)

$$\lim_{x\to\infty} \frac{3x}{2x+1} = \frac{3}{2}, \quad \epsilon = 0.005 \tag{3}$$

Solution: : We first work this out for general ϵ: Using the properties of the modulus, one has

$$\left|\frac{3x}{2x+1} - \frac{3}{2}\right| < \epsilon \Leftrightarrow -\epsilon < \left|\frac{3x}{2x+1} - \frac{3}{2}\right| < \epsilon$$

Now, we can add $3/2$ on all sides. Then, since we are interested in the limit where x becomes large and positive, we can multiply by the positive quantity $2x+1$ on all sides of the inequality to obtain

$$\left(\frac{3}{2} - \epsilon\right)(2x+1) < 3x < \left(\frac{3}{2} + \epsilon\right)(2x+1)$$

Subtracting $3x$ on all sides gives then 2 inequalities: one is always satisfied as it says that x has to be larger than some negative number. The other one gives the condition

$$x > \frac{1}{2\epsilon}\left(\frac{3}{2} - \epsilon\right) = \frac{3}{4\epsilon} = \frac{1}{2}.$$

For $\epsilon = 0.005$ this implies $x > 299/2$, and $N = 299/2$ will therefore do the trick.

(c)

$$\lim_{x\to\infty} \frac{8x+1}{2x+5} = 4, \quad \epsilon = 0.1 \tag{4}$$

This is completely analogous to the previous calculation: starting from

$$-\epsilon < \frac{8x+1}{2x+5} - 4 < \epsilon$$

one obtains the condition

$$x > \frac{19}{2\epsilon} - 5/2$$

which evaluates to $x > 95 - 5/2 = 185/2$ for $\epsilon = 0.1$. Hence $N = 185/2$ is good enough in this case.

3. (2) Determine whether the following functions are differentiable and, if so, calculate their derivative functions for all points x in their natural domains:

$$f(x) = x^2, \qquad f(x) = 3x, \tag{5}$$

Solution: Both functions are differentiable.

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0}(2x+h) = 2x, \tag{6}$$

$$\lim_{h\to 0} \frac{3(x+h) - 3x}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3. \tag{7}$$