**MA1S11 Calculus, Tutorial Sheet 3, Solutions**[1]

26-28 October 2011

## Questions

The numbers in brackets give the numbers of marks available for the question.

1. (2) Evaluate the limits

$$\lim_{x\to 3}\frac{(x+2)(x-3)}{x-3}, \qquad \lim_{x\to 1}\frac{x^2+4x-5}{x-1}, \qquad \lim_{x\to 2}\frac{x^2+4x-5}{x-1}. \tag{1}$$

*Solution:* We have

$$\lim_{x\to 3}\frac{(x+2)(x-3)}{x-3} == \lim_{x\to 3}(x+2) = 5, \tag{2}$$

and

$$\lim_{x\to 1}\frac{x^2+4x-5}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+5)}{x-1} = \lim_{x\to 1}(x+5) = 6, \tag{3}$$

Note that we may simplify the ratios under the limit symbol as $x$ gets as close as we like to 3 or 1 respectively, but remains different from these values. Finally,

$$\lim_{x\to 2}\frac{x^2+4x-5}{x-1} = \lim_{x\to 2}\frac{(x-1)(x+5)}{x-1} = 7 \tag{4}$$

2. (3) Consider the function $f(x) = 2(x+2)$. Find, for each of the given values $\varepsilon > 0$, a $\delta > 0$ such that for $|x-1| < \delta$ one has $|f(x) - 6| < \varepsilon$.

   a) $\varepsilon = 0.1$
   b) $\varepsilon = 0.01$
   c) $\varepsilon = 0.001$

*Solution:* We first consider the inequality $|f(x) - 6| < \varepsilon$. We have $|f(x) - 6| = |2x-2| = 2|x-1|$. Therefore, if $|x-1| < \varepsilon/2$ we have $|2x-2| < \varepsilon$. Therefore, with $\delta = \varepsilon/2$ we have found a $\delta > 0$ for *any* $\varepsilon > 0$ and thus proved that $\lim_{x\to 1} f(x) = 6$. In particular for the choices of $\varepsilon$ we have a) $\delta = 0.05$, b) $\delta = 0.005$, c) $\delta = 0.0005$.

3. (3) Consider the function $f(x) = x^2+2$. Determine the secant lines which go through the point $P(1,3)$ and any other point $Q(x_0, f(x_0))$ of the graph. Approximate the slope of the tangent line to the curve which passes through $P(1,3)$, by calculating the slope of the secant lines for points $Q(x_0, f(x_0))$ approaching $P$ (consider 3-4 numerical values). What is the equation for the tangent line in $P(1,3)$?

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*Hint*: the secant line must be a linear function of the form $y = m_{\text{sec}}x + b$, and $P(1,3)$ must be on its graph. This information is sufficient to determine $b$ in terms of $m_{\text{sec}}$, and you just need to calculate the slope $m_{\text{sec}}$ of the line through $P$ and $Q$.

*Solution:* Inserting $(x,y) = (1,3)$ for the point $P$ into the secant equation gives

$$3 = m_{\text{sec}} + b \quad \Rightarrow \quad b = 3 - m_{\text{sec}}$$

and determines $b$ in terms of $m_{\text{sec}}$. Next we use the definition of the slope of a line to obtain

$$m_{\text{sec}} = \frac{3 - f(x_0)}{1 - x_0} = \frac{3 - x_0^2 - 2}{1 - x_0} = \frac{1 - x_0^2}{1 - x_0} = \frac{(1 - x_0)(1 + x_0)}{1 - x_0}$$

Since the secant line is only defined for $P \neq Q$, i.e. $x_0 \neq 1$, we can divide by $1 - x_0$ and obtain $m_{\text{sec}} = 1 + x_0$. We may approach the point $P(1,3)$ from the left by choosing $x_0 = 0.9, 0.99$ which yields $m_{\text{sec}} = 1.9, 1.99$, or from the right by choosing $x_0 = 1.1, 1.01$ which gives $m_{\text{sec}} = 2.1, 2.01$. The slope of the tangent line is obtained as $m_{\tan} = \lim_{x_0 \to 1} m_{\text{sec}} = 2$, so that the equation for the tangent line in $P(1,3)$ is

$$y = 2x + 3 - 2 = 2x + 1.$$