MA1S11 Calculus, Tutorial Sheet 2, Solutions[1]

19-21 October 2011

Questions

The numbers in brackets give the numbers of marks available for the question.

1. (2) We define the function $f(x) = \sqrt{x^3 - 1}$. What is its natural domain? Is this function invertible? If so, what is its inverse function? Also give its domain and range?

   Solution: The function is real if the expression under the root is non-negative, i.e. if $x^3 - 1 \geq 0$ This simplifies to $x \geq 1$. The function is invertible, as it is one-to-one, it is strictly monotonically increasing. A graph of the function is given in figure 1

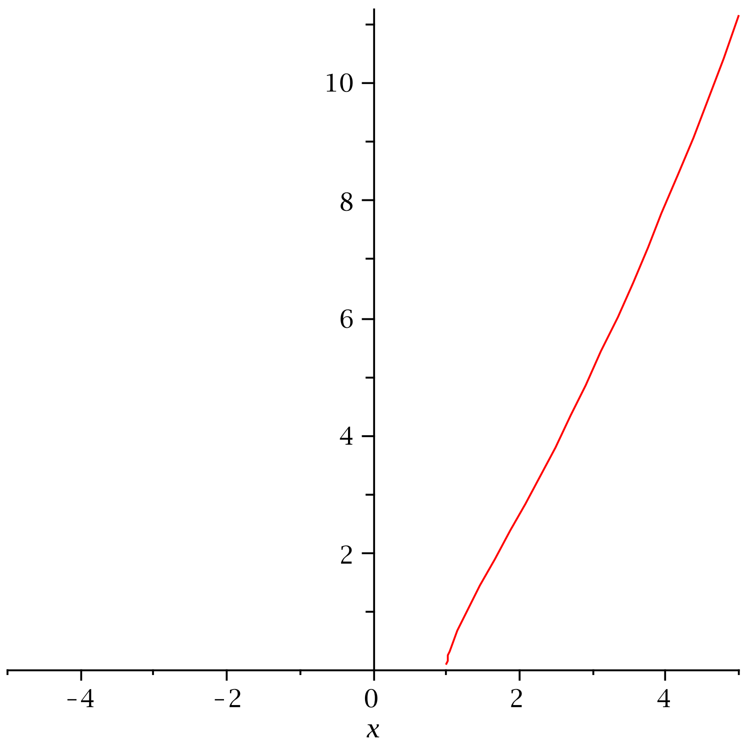

Figure 1: question 1: graph of $f(x) = \sqrt{x^3 - 1}$.

$$y = \sqrt{x^3 - 1} \Leftrightarrow y^2 + 1 = x^3 \Leftrightarrow x = \sqrt[3]{y^2 + 1}$$

Exchanging x and y we have

$$f^{-1}(x) = \sqrt[3]{x^2 + 1} = (x^2 + 1)^{1/3}$$

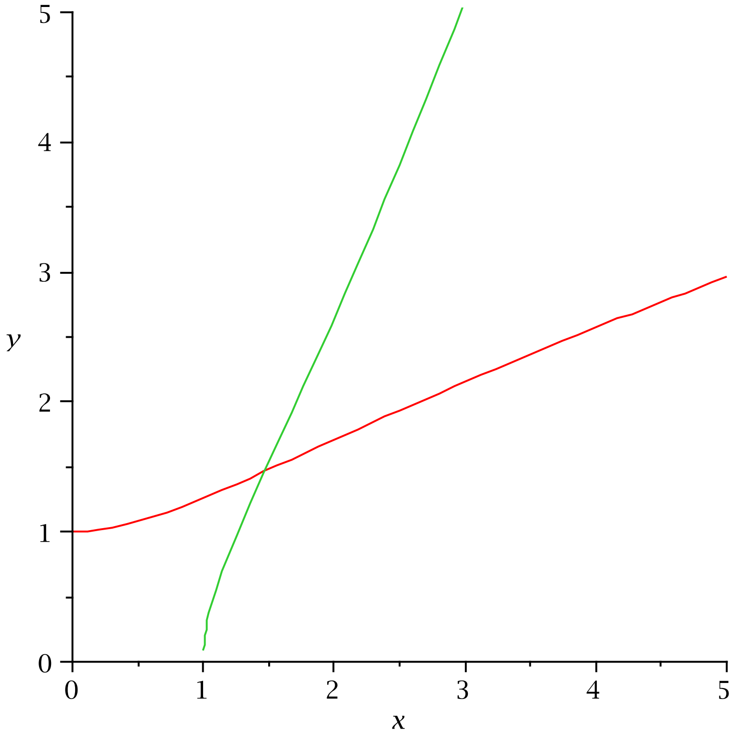

Figure 2: question 1: graph of $f(x) = \sqrt{x^3 - 1}$ and its inverse $f^{-1}(x) = .(x^2 + 1)^{1/3}$.

2. (3) Determine the symmetry properties (symmetry about x-axis, about y-axis, about the origin or none at all) of the following curves in the xy plane (you do not need to draw the graphs!):

$$y^2 = 3x^2, \qquad y^3 = x^7 + \sin(x), \qquad x^2 + \frac{y^2}{4} = 1, \qquad y^4 = x^{-3} + x.$$

Solution: We just need to see whether either a separate or a combined change of sign of x, y alters the equation or not. If not we have a symmetry. For instance $y^2 = 3x^2$ does not change for either $x \to -x$ or $y \to -y$ or both, so it is symmetric about the x-axis, about the y-axis and about the origin! In the case $y^3 = x^7 + \sin(x)$ we see that both sides are odd functions, i.e. if we replace $y \to -y$ and $x \to -x$ simultaneously the minus signs cancel and the equation doesn't change. Hence we have symmetry about the origin. The equation $x^2 + \frac{y^2}{4} = 1$ does not change whatever we do to the signs of x or y, so it has again all symmetries (about x-, y-axis and origin). In the last case we get the same equation for $y \to -y$, so we have a symmetry about the x-axis.

[1]Stefan Sint, `sint@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~sint/1S11.html`

3. (3) Consider the functions

$$f(x) = 3x + 1, \qquad g(x) = 2\sqrt{x+1}.$$

What are their natural domains and their ranges? Determine their inverse functions $f^{-1}(x)$ and $g^{-1}(x)$. and graph them together with f and g.

Solution: The natural domain for $f(x)$ are the real numbers. Letting x vary across the real numbers $f(x)$ covers all real numbers, so the range of f is the real numbers, too. For g there is a square root, so the expression under the square root must be non-negative, i.e. $x + 1 \geq 0$ which is to say $x \geq -1$. This defines the domain of g. As for its range, we see that the square root assumes all non-negative real numbers if x is varied over the domain, so the range is the non-negative real numbers. The inverses exist in both cases and are given by

$$f^{-1}(x) = \frac{1}{3}(x-1), \qquad g^{-1}(x) = \frac{x^2}{4} - 1.$$

The graphs are shown in the figures 2 and 3.

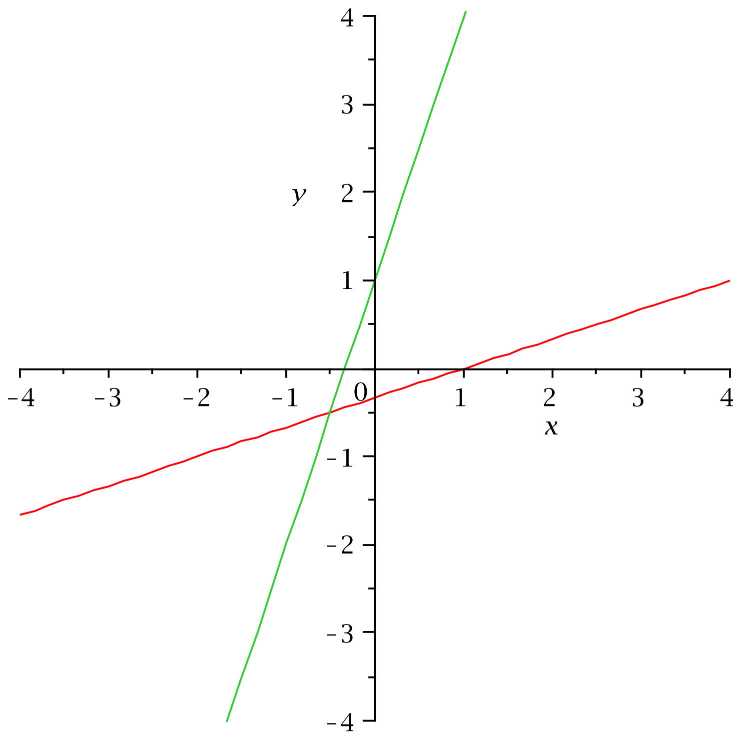

Figure 3: question 3: graphs of $f(x)$ and $f^{-1}(x)$.

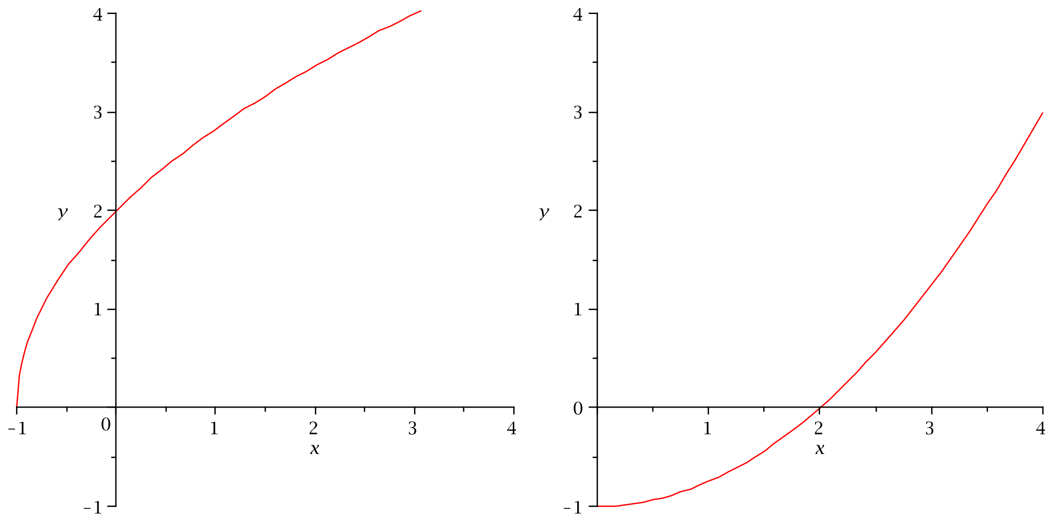

Figure 4: question 3: graphs of $g(x)$ and $g^{-1}(x)$.