1S11 Tutorial Sheet 1: Solutions[1]

12-14 October 2011

Questions

1. (1) What is the natural domain and the range of $f(x) = \sqrt{x^2 - 2x - 3}$, where you can note the factorization: $x^2 - 2x - 3 = (x + 1)(x - 3)$.

   Solution: The natural domain covers all the places where the function is real. The function here is real provided the term under the square root is not negative. Taking into account the factorisation this is the case if both factors are either negative or zero, i.e. $x \le -1$ or if both factors are either positive or zero, $x \ge 3$ so the natural domain is $\{x : x \le -1 \,\mathrm{or}\, x \ge 3\}$

2. (1) What is the natural domain of $f(x) = \sqrt{x + 1}/(x^2 - 1)$?

   Solution: This is a ratio of 2 functions. The natural domain for the numerator function is $x \ge -1$, the natural domain for the denominator are the real numbers. When forming the ratio the natural domain is the intersection of the domains for numerator and denominator function, minus the zeros of the denominator (here at $x = \pm 1$). So the domain for the ratio is $\{x : x > -1,\, x \ne 1\}$

3. (1) Suppose f and g are odd functions. Show that their sum $f + g$ is an odd function and that their product fg is an even function.

   Solution: Define the sum function $h(x) = f(x) + g(x)$. Now $h(-x) = f(-x) + g(-x) = -f(x) - g(x) = -h(x)$, so $h = f + g$ is odd. Similarly, setting $h(x) = f(x)g(x)$ one has $h(-x) = f(-x)g(-x) = -f(x)[-g(x)] = h(x)$, so fg is even.

4. (1) Determine whether each of the following is even or odd or neither and show why: $f(x) = 1/x^3$, $f(x) = \cos(x + 1)$, $f(x) = \sin x^3$, $f(x) = x^4 + 5$

   Solution: $1/x^3$, $\sin(x^3)$ are odd and $x^4 + 5$ is even. As usual this is checked by inserting $-x$ for x and seeing whether $f(-x) = \pm f(x)$. For $f(x) = \cos(x + 1)$ we have

   $$f(-x) = \cos(-x + 1) \ne \pm f(x), \tag{1}$$

   so this function is neither even nor odd.

5. (2) For any function $f(x)$ whose domain is the real numbers show $g(x) = f(x)+f(-x)$ is even and $h(x) = f(x) - f(-x)$ is odd. Show that any function whose domain is the real numbers can be written as the sum of an even and an odd function. Work out g and h for $f(x) = x^2 - x + 1$.

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Solution: $g(-x) = f(-x) + f(x) = g(x)$ so g is even, while $h(-x) = f(-x) - f(x) = -h(x)$ is odd. We can decompose any function f as follows,

$$f(x) = \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) = \frac{1}{2}(g(x) + h(x)), \tag{2}$$

so g and h are, up to the factor $1/2$, the even and odd parts of f. For the example of $f(x) = x^2 - x + 1$ we then have

$$\begin{aligned} g(x) &= f(x) + f(-x) = x^2 - x + 1 + (-x)^2 - (-x) + 1 = 2x^2 + 2, \\ h(x) &= f(x) - f(-x) = x^2 - x + 1 - (-x)^2 + (-x) - 1 = -2x, \end{aligned}$$

so the even and odd parts of this function are $x^2 + 1$ and $-x$, respectively.

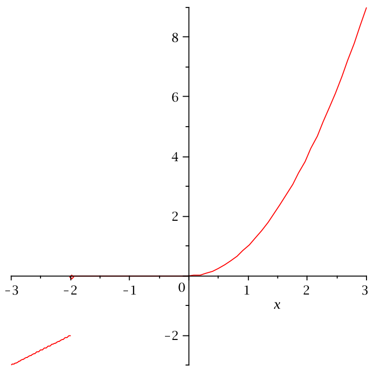
6. (2) Consider the piecewise defined function

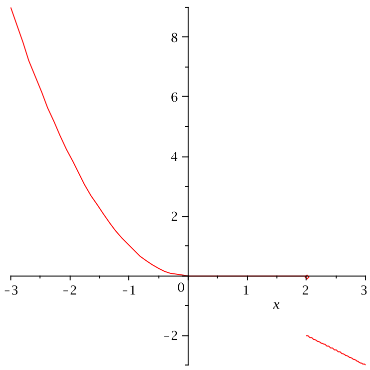
$$f(x) = \begin{cases} x & x < -2 \\ 0 & -2 \leq x < 0 \\ x^2 & x \geq 0 \end{cases}$$

   (a) (1) Sketch the graph of $f(x)$, $f(-x)$ and $-f(x)$.

   (b) (1) Sketch the graph of $f(x) + 1$, $f(x + 1)$, $f(2x)$ and $2f(x)$.

   Solution: See figure 1 for part a) and figures 2 and 3 for part b)





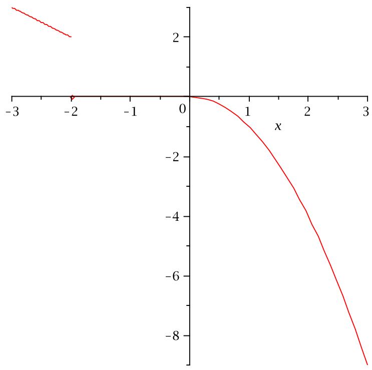


Figure 1: Question 6a: graphs of $f(x)$, $f(-x)$ and $-f(x)$.

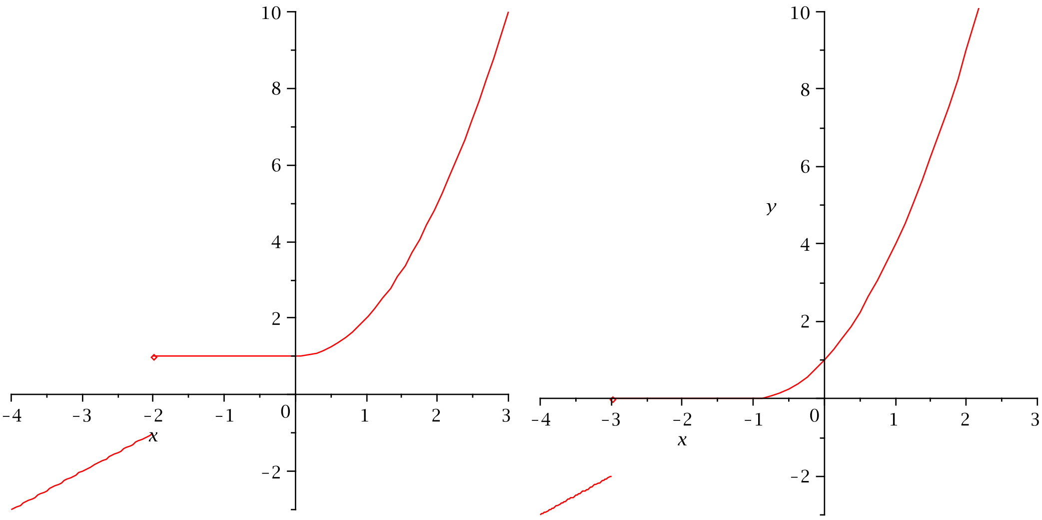


Figure 2: Question 6b): graphs of $f(x) + 1$, $f(x + 1)$.

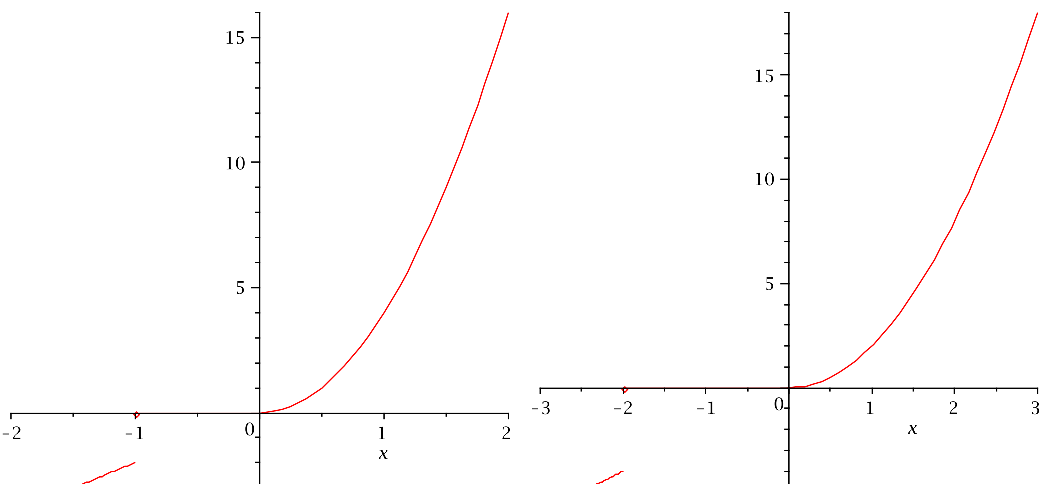


Figure 3: Question 6b): graphs of $f(2x)$ and $2f(x)$.