MA1S11 Tutorial Sheet 7[1]

30 November - 1 December 2011

Useful facts:

- **Trigonometric functions**

$$\begin{aligned}\frac{d}{dx}\cos x &= -\sin x \\ \frac{d}{dx}\sin x &= \cos x \end{aligned} \tag{1}$$

- **Chain rule**: With $u = g(x)$

$$\frac{d}{dx} f(g(x)) = \frac{df}{du}\frac{du}{dx} = f'(u)g'(x). \tag{2}$$

- **Implicit differentiation**: Example: find dy/dx for $y^2 + x^2 = 1$, this gives a relationship between x and y without expressing y explicitely as a function of x. Differentiate across

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}1 = 0 \tag{3}$$

so

$$2x\frac{dx}{dx} + \frac{dy^2}{dx} = 0 \tag{4}$$

and hence

$$2x + 2y\frac{dy}{dx} = 0 \tag{5}$$

or, solving for dy/dx

$$\frac{dy}{dx} = -\frac{x}{y} \tag{6}$$

- **Relative Extrema** A point $x = x_0$ is a relative maximum if there is an open interval (a, b) containing x_0 such that $f(x) \leq f(x_0)$ for all $x \in (a, b)$; it is a relative minimum if $f(x) \geq f(x_0)$ for all $x \in (a, b)$.

- **Critical points** If $f'(x_0) = 0$ or if f is not differentiable at $x = x_0$ then x_0 is a critical point of f. To find out whether f has a relative extremum at x_0 one may determine whether $f'(x)$ changes sign across x_0. If so f has a relative extremum at x_0 which is a relative maximum for a plus-minus change and a relative minimum for minus-plus change of $f'(x)$.

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- **Second derivative test** If f is twice differentiable one may determine whether a critical/stationary point x_0 corresponds to a relative maximum or minimum of f by computing f''. If $f''(x_0) > 0$ then f has a relative minimum at x_0, if $f''(x_0) < 0$ then f has a relative maximum at x_0. If $f''(x_0) = 0$ it is undecided, it could be a relative extremum or it could be a **point of inflection**, i.e. a point joining a **concave up** $f''(x) > 0$ and a **concave down** $f''(x) < 0$ interval.

Questions

The numbers in brackets give the numbers of marks available for the question.

1. (4) Determine the derivative of the inverse function f^{-1}, by using the chain rule and implicit differentiation of the equation

$$f\left(f^{-1}(x)\right) = x \tag{7}$$

Then use the result to obtain the derivative of the inverse functions of $f(x) = \sqrt{x}$ ($x \geq 0$) and $f(x) = \cos x$ (for $x \in [0, \pi]$).

(Hint: for the inverse cosine function you'll need the identity $\cos^2 x + \sin^2 x = 1$)

2. (4) Determine the zeros, the relative maxima and minima, the regions of concavity up/down and the limit behaviour $x \to \pm\infty$ for

$$f(x) = (x+1)^2(x-1) = x^3 + x^2 - x - 1 \tag{8}$$

Then draw the graph of f using all the gathered information.