MA1S11 Tutorial Sheet 5[1]

16-18 November 2011

Useful facts:

- The **derivative**
$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

- **Higher derivatives**:
$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) = \frac{d^2}{dx^2} f(x), \\ f'''(x) &= \frac{d}{dx} f''(x) = \frac{d^2}{dx^2} f'(x) = \frac{d^3}{dx^3} f(x), \end{aligned} \tag{2}$$
and so on, a bracket number is used after f''', for example f'''' is written $f^{(4)}$.

- **Constant function**: If $f(x) = c$ (with a constant c) then $f'(x) = 0$. Conversely, if $f'(x) = 0$ then $f(x) = c$ with some constant c.

- **Power rule**: If $f(x) = x^n$ then $f'(x) = nx^{n-1}$. In class we have proved this for positive integers, but it also holds for negative integers n.

- **Linearity of differentiation (c a real number)**:
$$(cf)' = cf', \qquad (f+g)' = f' + g'. \tag{3}$$

- **Product rule**:
$$(fg)' = fg' + f'g. \tag{4}$$

- **Trigonometric functions**
$$\frac{d}{dx} \cos x = -\sin x, \qquad \frac{d}{dx} \sin x = \cos x, \tag{5}$$

- **Binomial formula** For a natural number $n \geq 0$
$$\begin{aligned} (a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{2} a^2 b^{n-2} + nab^{n-1} + b^n \end{aligned} \tag{6}$$

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where the binomial coefficients are defined by

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n \\ n-k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \qquad \begin{pmatrix} n \\ 0 \end{pmatrix} = 1 \tag{7}$$

The binomial coefficients can also be inferred from Pascal's triangle (cp. class of November 4).

Questions
The numbers in brackets give the numbers of marks available for the question.

1. (2) Work out f', f'' and keep going differentiating until you get zero.
$$f(x) = \frac{1}{8}x^8 + x^5 + 4x^2 - x + 4 \tag{8}$$
2. (2) Differentiate
$$f(x) = \frac{1}{x^2}, \tag{9}$$
and
$$f(x) = x + \frac{1}{x}. \tag{10}$$
3. (2) Prove the product rule $(fg)' = f'g + fg'$. (Hint: use the limit definition of the derivative for the product function $f(x)g(x)$ and add and subtract appropriate terms in the numerator such that you get $f'(x)g(x) + f(x)g'(x)$ in the limit $h \to 0$.)
4. (2) $f(x) = \sin^2 x + \cos^2 x$, we know by the Pythagorean theorem that $f(x) = 1$, but check that $f'(x) = 0$ from the original formula with sines and cosines. If $f'(x) = 0$ then $f(x)$ is a constant, check the value of the constant by working out $f(0)$. Is the answer consistent with the Pythagorean theorem?
5. (2) Iterate the product rule by working out the higher derivatives of a product, $(fg)''$, $(fg)'''$ and $(fg)^{(4)}$. Then use the notation
$$f = f^{(0)}, \quad f' = f^{(1)}, \qquad f'' = f^{(2)}, \ldots \tag{11}$$
and analogously for g. Can you discover a relation to the binomial formula (see "useful facts" section) and thus give an expression for the nth derivative $(fg)^{(n)}$?

Extra Question
The question is extra; you don't need to do it in the tutorial class.

1. Iterate the product rule by working out the higher derivatives of a product, $(fg)''$, $(fg)'''$ and $(fg)^{(4)}$. Then use the notation
$$f = f^{(0)}, \quad f' = f^{(1)}, \qquad f'' = f^{(2)}, \ldots \tag{12}$$
and analogously for g. Can you discover a relation to the binomial formula (see "useful facts" section) and thus give an expression for the nth derivative $(fg)^{(n)}$?