MA1S11 Calculus, Tutorial Sheet 4[1]

2-4 November 2011

Useful facts:

- **Infinite Limit** (Informal definition): If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a but not equal to a then we write

$$\lim_{x\to a} f(x) = \infty \tag{1}$$

  There are also minus infinity and one-sided versions of this, for example, if the value of $f(x)$ can be made as large a negative number as we like by taking values of x sufficiently close to a and greater than a then we write

$$\lim_{x\to a+} f(x) = -\infty \tag{2}$$

- **Limit at Infinity** (Informal definition): If the value of $f(x)$ can be made as close as we like to L by taking sufficiently large values of x then we write

$$\lim_{x\to\infty} f(x) = L \tag{3}$$

- **Limit at Infinity of Rational Functions**: Divide above and below by the highest power in the denominator. For instance,

$$f(x) = \frac{x^4+2}{2x^4+x^2+6} = \frac{1+2/x^4}{2+1/x^2+6/x^4} \tag{4}$$

  This is a ratio of functions which both have a finite limit as $x \to \infty$, so that

$$\lim_{x\to\infty}(1+2/x^4) = 1, \qquad \lim_{x\to\infty}(2+1/x^2+6/x^4) = 2 \quad \Rightarrow \quad \lim_{x\to\infty} f(x) = \frac{1}{2}$$

- **Continuous functions:** A function is continuous at x if the function is defined at x, $\lim_{h\to 0} f(x+h)$ exists and the limit coincides with the function value, i.e.

$$\lim_{h\to 0} f(x+h) = f(x). \tag{5}$$

- **Derivative**: A function f is differentiable at x if f is continuous at x and the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{6}$$

  exists. If this is the case the limit is denoted by $f'(x)$ ("f prime of x"). When taken as a function of x this defines a new function, the derivative function f'.

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Questions

The numbers in brackets give the numbers of marks available for the question.

1. (3) Evaluate the limits

$$\lim_{x\to\infty} \frac{5x^2+1}{2x^2+x-4}, \qquad \lim_{x\to\infty} \frac{4x^3+x^2}{x^3-x+1}, \qquad \lim_{x\to\infty} \frac{-3x^5+2x^3-2}{4x^5+x^3+4}. \tag{7}$$

2. (3) Given $\epsilon > 0$ and the limit L of the function $f(x)$ for $x \to \infty$, find $N > 0$ such that $|f(x) - L| < \epsilon$ if $x > N$:

   (a)
   $$\lim_{x\to\infty} \frac{1}{(x+2)^2} = 0, \quad \epsilon = 0.01 \tag{8}$$

   (b)
   $$\lim_{x\to\infty} \frac{3x}{2x+1} = \frac{3}{2}, \quad \epsilon = 0.005 \tag{9}$$

   (c)
   $$\lim_{x\to\infty} \frac{8x+1}{2x+5} = 4, \quad \epsilon = 0.1 \tag{10}$$

3. (2) Determine whether the following functions are differentiable and, if so, calculate their derivative functions for all points x in their natural domains:

$$f(x) = x^2, \qquad f(x) = 3x. \tag{11}$$