MA2321: Exercises 8, 2011

30 November 2011

1. Let $f : \mathbf{R}^{n\times n} \to \mathbf{R}$ be the function $f(A) = \det(A)$. Find $f'(A)$ and show that $f'(A)A = n \det A$. [Hint: write $\det A = D(a_1, a_2, \dots, a_n)$ where $a_1, a_2, \dots, a_n$ are the columns of A, and note that D is linear in each variable.]

2. Show that
$$\{A \in \mathbf{R}^{n\times n} | \det A = 1\}$$
is a manifold of dimension $n^2 - 1$.

3. Let $f = F(g^1, ..., g^k)$ where $(g^1, ..., g^k)$ are scalar fields and F is a C^∞ function of k independent variables. Show that:
$$df = \sum_{i=1}^{k} \frac{\partial F}{\partial x^i}(g^1, ..., g^k) dg^i$$

4. Let π be a surjective map of a topological space X onto a set Y. Call a subset V of Y *open in* Y if and only if $\pi^{-1}(V)$ is open in X. Show that this defines a topology on Y (caled the *quotient topology on* Y *with respect to the map* π).

5. Let π be the map from $\mathbf{R^3} - \mathbf{0}$ to the set $\mathbf{P^2}$ of 1-dimensional subspaces of $\mathbf{R^3}$ where $\pi(x, y, z) = [x, y, z]$ denotes the 1-dimensional subspace containing the point (x, y, z).

   $\mathbf{P^2}$, with the quotient topology, is called the *real projective plane.*

   Show that if α is a non-zero scalar then $[\alpha x, \alpha y, \alpha z] = [x, y, z]$.

   Show that the 1-dimensional subspace $[x, y, z]$ intersects the plane $z = 1$ at the point $(x/z, y/z, 1)$ if $z \neq 0$.

   Show that $(x/z, y/z)$ is a 2-dimensional coordinate system on $\mathbf{P^2}$ with domain $z \neq 0$.

   Show that the coordinates on $x \neq 0$, $y \neq 0$ and $z \neq 0$ respectively are mutually C^∞ compatible.