MA2321: Exercises 4, 2009

18 November 2009

1. Find a sufficient condition that the functions

$$u = x + y \text{ and } v = xy$$

should be a coordinate system with domain an open neighbourhood of (a, b) in $\mathbf{R}^2$.

2. Find a sufficient condition such that the equation

$$xyz^2 + yzx^2 + zxy^2 = 1$$

should define z as a C^∞ function of x and y on an open neighbourhood of (a, b, c), where (a, b, c) is a solution.

3. Let X be the subset of $\mathbf{R}^5$ given by the equations

$$F(x, y, z, t, u) = 0$$

$$G(x, y, z, t, u) = 0$$

where F and G are C^∞ functions on an open set V in $\mathbf{R}^5$. Write down sufficient conditions for x, y, z to be coordinates on X at a given point (a, b, c, d, e) of X.

4. Show that the equation

$$xt + yz = 1$$

defines a 3-dimensional manifold in $\mathbf{R}^4$.

5. Let A be a real symmetric $n \times n$ matrix, let $k \neq 0$, and let $\mathbf{R}^n$ have its usual scalar product. Show that the equation

$$(Ax|x) = k$$

defines an $(n-1)$-dimensional manifold in $\mathbf{R}^n$.

6. Show that the quadric

$$3x^2 + 4y^2 + 5z^2 + 6xy + 8xz - 2yz = 4$$

is a manifold in $\mathbf{R}^3$.

7. Let f be the map from the space of real $n \times n$ matrices to the space of real symmetric $n \times n$ matrices, given by: $f(A) = A^t A$.

   Recall that if A is an orthogonal matrix then the linear operator $f'(A)$ is surjective.

   Deduce that the set of real orthogonal matrices is a manifold, and find its dimension.

8. Let $f : \mathbf{R}^{n\times n} \to \mathbf{R}$ be the function $f(A) = \det(A)$. Find $f'(A)$ and show that $f'(A)A = n \det A$. [Hint: write $\det A = D(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are the columns of A, and note that D is linear in each variable.]

9. Show that
   $$\{A \in \mathbf{R}^{n\times n} | \det A = 1\}$$
   is a manifold of dimension $n^2 - 1$.