MA2322 Exercises 3; 2012

31 January 2012

1. Let T be a tensor with components α^i_j with respect to a basis. Write down the components of the tensors which can be formed from T by tensor products and contractions.

2. Give an example of tensors S and T such that $S \otimes T$ is not equal to $T \otimes S$.

3. Let M be a finite dimensional real vector space . Two bases u_i and w_i are said to have the *same orientation* if the transition matrix P from u_i to w_i has positive determinant, and have *opposite orientation* if P has negative determinant. Show that "same orientation" is an equivalence relation on the set of bases for M, and that there are just two equivalence classes. M is said to be *oriented* if one of the equivalence classes is chosen and its members are called "positively oriented".

4. Let M be an oriented finite dimensional real vector space with positive definite symmetric scalar product (.|.).

   (a) Prove that the transition matrix P from one positively oriented orthonormal basis u_i to another positively oriented orthonormal basis w_i has determinant 1.

   (b) Prove that if M is n-dimensional there is a unique tensor vol (called the *volume form of M*) on M of degree n and all indices lower which is skew-symmetric and takes the value 1 on a given positively oriented orthonormal basis u_i.

   (c) Prove that vol takes the value 1 on every positively oriented orthonormal basis.

   (d) If w_i is a positively oriented basis show that the components G of (.|.) with respect to w_i are Q^tQ where Q is the inverse of the transition matrix P to w_i from a positively oriented orthonormal basis u_i.. Show that the value of vol on w_i is equal to the square root of the determinant of G.

5. Let (.|.) be a symmetric non-singular scalar product on a finite dimensional real vector space M with dual space M^*, and let L be the lowering operator. Let T be a tensor of type $M^* \times M^* \times M \rightarrow R$ and lower the second index to get a tensor L_2T of type $M^* \times M \times M \rightarrow R$ defined by: $L_2T(f, y, x) = T(f, Ly, x)$. Find the components of L_2T in terms of the components of T and of (.|.).