MA2321 Exercises 3; 2009

27 October 2009

1. Show that if f is a C^2 function of one variable then $u = f(x - ct)$ is a solution of the equation
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

2. Let $u = f(3x, x^2 + x^3, 5)$ where f is a C^1. function of three independent variables. Find $\frac{du}{dx}$.

3. Let f and g be functions of two real variables and suppose that
$$f(x, y) = F(g(x, y))$$
where F is a function of one real variable. Assume that the functions f, g, F are all differentiable. Prove that the Jacobian:
$$\frac{\partial(f, g)}{\partial(x, y)}$$
is zero.

4. On each tangent space of R^3 define a scalar product such that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are orthonormal.

   Let f be a differentiable real valued function on R^3, and X be the surface with equation $f(x, y, z) = c$ where c is a constant. Show that the velocity vector of a parametrised path $\alpha(t)$ in X at parameter t is orthogonal to the *gradient of* f :
$$\nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$$
at $\alpha(t)$.

5. Let f and g be a differentiable real valued functions on R^3. Denote by fdg the function on the tangent bundle of R^3 whose value on a tangent vector v at at point a is $< fdg, v >= f(a)vg$. Prove that:
$$dg = \frac{\partial g}{\partial x}(a)dx + \frac{\partial g}{\partial x}(a)dy + \frac{\partial g}{\partial z}(a)dz$$
[Evaluate both sides on the velocity vector of a parametrised path α] Note that dg is called the *differential* of g and represents the rate of change of g.

6. Let f and g be diffeerntiable real valued functions on R^3. Prove that $d(fg) = fdg + gdf$. [Evaluate both sides on the velocity vector of a parametrised path α]

7. let $g^1, g^2, ..., g^k$ be differentiable functions on an open set V in R^3. Let $f = F(g^1, g^2, ..., g^k)$ where F is a differentiable function on R^k. Prove that:

$$df = \frac{\partial F}{\partial x^1}[g^1, g^2, ..., g^k]dg^1 + ... + \frac{\partial F}{\partial x^k}[g^1, g^2, ..., g^k]dg^k$$

8. Let x and y be the usual coordinate functions on $\mathbf{R^2}$ and r and θ the functions given by $x = r\cos\theta, y = r\sin\theta$ on the set $-\pi < \theta < \pi$,and $r > 0$. Let $\frac{\partial}{\partial r}$ be the velocity vector of the path $\theta =$ constant, parametrised by r. Let $\frac{\partial}{\partial \theta}$ be the velocity vector of the path $r =$ constant, parametrised by θ

   (a) Show that $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ are a basis for the tangent space at each point of $-\pi < \theta < \pi, r > 0$.

   (b) Write $\frac{\partial}{\partial x}$ as a linear combination of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$.

   (c) Write $\frac{\partial}{\partial r}$ as a linear combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

   (d) Let f be a real valued differentiable function on $\mathbf{R^2}$. Show that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$$

   (e) Show that the value of $(dx)^2 + (dy)^2$ on a tangent vector v is the square of the norm of v

   (f) Write $(dx)^2 + (dy)^2$ in terms of dr and $d\theta$.

9. Calculate:

$$(dx)^2 + (dy)^2 + (dz)^2$$

in terms of the *spherical polar coordinates* r, θ, ϕ where

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$