# MA2322 Exercises 2; 2012

23 January 2012

1. Let $M$ be an n-dimensional real vector space. Let $T$ be a non-zero skew-symmetric tensor on $M$ of degree n and all indices lower.

   Show that the vectors $x_1, ...., x_n$ are linearly independent if and only if $T(x_1, ...., x_n)$ is non-zero.

   If $S$ is a linear operator on $M$ with matrix $A$ with respect to a basis $u_1, ...., u_n$, show that for any vectors $x_1, ...., x_n$ we have $T(Sx_1, ...., Sx_n)$ is equal to $T(x_1, ...., x_n)$ multiplied by the determinant of the matrix $A$. [Hint: both sides are multilinear functions of $x_1, ...., x_n$, so check if they agree on $u_1, ...., u_n$ ]

   [Note: this shows that we can define the determinant of $S$ to be the determinant of $A$ and this is independent of the choice of basis $u_1, ...., u_n$]

2. Show that there is a unique tensor $D$ of degree $n$ on $R^n$ with all indices lower which is skew-symmetric and takes the value 1 on the usual basis $e_1 = (1, 0, 0, .....0), ....., e_n = (0, 0, 0, .....1)$

   Show that if $A$ is an $n \times n$ matrix then $D(Ae_1, ...., Ae_n)$ is the determinant of $A$

   Show that the value of $D$ on $(a_1, ...., a_n)$ is the determinant of the matrix with columns $(a_1, ...., a_n)$

3. Show that if $A$ and $B$ are $n \times n$ matrices then the determinant of $AB$ is equal to the determinant of $A$ times the determinant of $B$. [Hint: use the previous question]

4. Show that an $n \times n$ matrix $A$ is invertible if and only if its determinant is non-zero.

5. If $x$ is a vector in $R^n$ show that its ith component with respect to the basis $e_1 = (1, 0, 0, .....0), ....., e_n = (0, 0, 0, .....1)$ is equal to $D(e_1, ...., x, ..., e_n)$ where $x$ is placed in the ith slot.

6. Let $Ax = b$ where $A$ is an invertible $n \times n$ matrix, and $x$ and $b$ are vectors in $R^n$ written as column matrices. Use the previous question to find the ith component of $x$ as a ratio of determinants (Cramers Rule) and find the (i,j) entry of $A^{-1}$ as a ratio of determinants (Formula for $A^{-1}$ )