MA2322 Exercises 1; 2012

17 January 2012

1. Let $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ be the usual line element on $\mathbf{R^3}$ where x ,y, z are the usual coordinate functions. Let f be a scalar field on open V in $\mathbf{R^3}$.

   (a) If α is a parametrised path along which f is constant, show that grad$f$ at $\alpha(t)$ is orthogonal to the velocity vector at $\alpha(t)$.

   (b) Let a be a point of the surface X with equation $f(x, y, z) = 0$ at which $\frac{\partial f}{\partial z}$ is non-zero. By pulling back df to X, write dz as a linear combination of dx and dy on a neighbourhood of a in X when x ,y, z are restricted to X.

   (c) If $z = g(x, y)$ on a neighbourhood of a in X, write down $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in terms of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

   (d) Pull back the line element to X and find its components with respect to coordinates x,y on a neighbourhood of a in X

2. Let $(.|.)$ be a symmetric non-singular scalar product on a finite dimensional real vector space M with dual space M^*.

   Define a linear operator $L : M \to M^*$, called *lowering the index* by:

   $$< Lx, y >= (x|y)$$

   for all vectors x and y in M.

   (a) Show that L has kernel $\{0\}$.

   (b) Show that L is an vector space isomorphism.

   (c) Show that the matrix $g_{ij} = (u_i|u_j)$ of the scalar product with respect to the basis u_i is also the matrix of the linear operator L with respect to the basis u_i for M and the dual basis u^i for M^*:
   $$Lu_i = g_{ij}u^j$$

   (d) The inverse R of the operator L is called *lowering the index*. Show that

   $$Ru^i = g^{ij}u_j$$

   where g^{ij} is the matrix inverse of g_{ij}.

(e) If $x \in M$ then $\alpha^i =< u^i, x >$ are called the *contravariant components* of x, and $\alpha_i = (x|u_i)$ are called the *covariant components* of x. Show that

$$\alpha_i = g_{ij}\alpha^j$$

$$\alpha^i = g^{ij}\alpha_j$$

and that the covariant components of x are the components of Lx.

(f) Define a scalar product on M^* by:

$$(f|h) = (Rf|Rh)$$

Show that the components of the scalar product on M^* are g^{ij} with respect to the dual basis u^i.