Energy-Momentum Tensor in EM.

Have $$T^{\beta\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\lambda)} \partial^\beta A^\lambda - g^{\alpha\beta}\mathcal{L}$$

and by calculation e.g.

$$\left.\begin{aligned}
T^{00} &= \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) + \frac{1}{4\pi}\nabla\cdot(\Phi\vec{E}) \\
T^{i0} &= \frac{1}{4\pi}(\vec{E}\times\vec{B})_i + \frac{1}{4\pi}\nabla\cdot(A_i\vec{E}) \\
T^{0i} &= \frac{1}{4\pi}(\vec{E}\times\vec{B})_i + \frac{1}{4\pi}\left[(\nabla\times\phi\vec{B})_i - \frac{\partial}{\partial x}(\phi E_i)\right]
\end{aligned}\right\} T^{\beta\alpha} \text{ not symmetric}$$

Note that $\partial_\alpha T^{\beta\alpha} = 0$ but the lack of symmetry means quantities like $M^{\alpha\beta\gamma} = T^{\beta\alpha}x^\gamma - T^{\gamma\alpha}x^\beta$ not conserved ie $\partial_\alpha M^{\alpha\beta\gamma} \neq 0$. (only true if T is symm.)

But definition of $\mathbb{T}$ is not unique – up to term $\sim \frac{\partial}{\partial x^\lambda}\varphi^{\mu\nu\lambda}$

with $\varphi^{\mu\nu\lambda} = -\varphi^{\mu\lambda\nu}$

and $$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{\partial}{\partial x^\lambda}\varphi^{\mu\nu\lambda}$$

is the symmetric, (canonical) stress-energy / energy-momentum tensor.

add a term $\sim \frac{\partial}{\partial x^\lambda} \varphi^{\mu\nu\lambda}$ with $\varphi^{\mu\nu\lambda} = -\varphi^{\mu\lambda\nu}$

so that $\qquad \tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{\partial}{\partial x^\lambda} \varphi^{\mu\nu\lambda}$

ensuring $\qquad \partial_\nu \tilde{T}^{\mu\nu} = 0 \qquad \left(\text{since } \frac{\partial^2}{\partial x^\nu \partial x^\lambda} \varphi^{\mu\nu\lambda} = 0 \right)$

[Note there is no change to total 4-mom p^μ

since
$$p^\mu = \frac{1}{c} \int T^{\mu 0} d^3x$$
$$= \frac{1}{c} \int T^{\mu\nu} dS_\nu \quad , \quad \text{constant } x^0 \text{ surface}$$

$$\Rightarrow \int \frac{\partial \varphi^{\mu\nu\lambda}}{\partial x^\lambda} dS_\nu$$ $\longrightarrow$ can show integrals of this type are zero]

$\Rightarrow$ can add this term to $T^{\mu\nu}$ for E&M so it becomes symm. in indices

Now what is $\frac{\partial \varphi^{\mu\nu\lambda}}{\partial x^\lambda}$?

recall,
$$T^{\beta\alpha} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda + \frac{g^{\alpha\beta}}{4\pi} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{F_{\kappa\rho} F^{\kappa\rho} = \mathcal{L}_{EM}}$$

now, add
$$\frac{1}{4\pi} \frac{\partial A^\beta}{\partial x_\lambda} F^\alpha{}_\lambda = \frac{1}{4\pi} \frac{\partial}{\partial x_\lambda} (A^\beta F^\alpha{}_\lambda) - \frac{1}{4\pi} A^\beta \frac{\partial F^\alpha{}_\lambda}{\partial x_\lambda}$$

and using e.o.m
$$\partial^\lambda F^\alpha{}_\lambda = 0 \quad = \frac{1}{4\pi} \partial^\lambda (A^\beta F^\alpha{}_\lambda)$$
$$= \frac{1}{4\pi} \partial_\lambda (A^\beta F^{\alpha\lambda})$$

ie $\qquad \varphi^{\beta\alpha\lambda} = \frac{A^\beta F^{\alpha\lambda}}{4\pi} = -\varphi^{\beta\lambda\alpha}$

Traditionally write a symmetric stress-eng. tensor

$$T^{\beta\alpha} = \Theta^{\beta\alpha} + \partial_\lambda \varphi^{\beta\alpha\lambda}$$

so

$$\Theta^{\beta\alpha} = -\frac{1}{4\pi}\left(F^\alpha{}_\lambda \partial^\beta A^\lambda - F^\alpha{}_\lambda \partial^\lambda A^\beta\right) + \frac{g^{\alpha\beta}}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

$$= \frac{1}{4\pi}\left(-F^\alpha{}_\lambda F^{\beta\lambda} + \frac{g^{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

$$\left(\text{using } \partial^\lambda \varphi^{\beta\alpha\lambda} = \partial^\lambda (A^\beta F^{\alpha}{}_{\lambda}) = F^\alpha{}_\lambda \partial^\lambda A^\beta + \underbrace{A^\beta \partial^\lambda F^\alpha{}_\lambda}_{=0}\right)$$

and this Θ is symm. & traceless : $\Theta^\alpha{}_\alpha = 0$.

AND

$$\Theta^{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$$

$$\Theta^{0i} = \frac{1}{4\pi}(\vec{E} \times \vec{B})$$

$$\Theta^{ij} = -\frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\right)$$

and

$$\partial_\alpha \Theta^{\alpha\beta} = 0$$

Verify in this case cons. of ang mom ie $M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^\gamma - \Theta^{\alpha\gamma} x^\beta$.

Finally

Sources: calculate $\partial_\mu \Theta$

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{4\pi}\left[\partial^\mu (F_{\mu\lambda} F^{\lambda\nu}) + \frac{1}{4}\partial^\nu (F_{\mu\lambda} F^{\mu\lambda})\right]$$

$$= \frac{1}{4\pi}\left[(\partial^\mu F_{\mu\lambda}) F^{\lambda\nu} + F_{\mu\lambda}\partial^\mu F^{\lambda\nu} + \frac{1}{2} F_{\mu\nu}\partial^\nu F^{\mu\nu}\right]$$

and recall $\partial^\mu F_{\mu\lambda} = \frac{4\pi}{c} J_\lambda$

$$\Rightarrow \partial_\mu \Theta^{\mu\nu} = \frac{1}{4\pi}\left[\frac{4\pi}{c} J_\lambda F^{\lambda\nu} + \frac{F_{\mu\lambda}}{2}\left(\partial^\mu F^{\lambda\nu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda}\right)\right]$$

and the HME : $\varepsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0 \Rightarrow (\alpha = 0)$

$$\partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} + \partial^\lambda F^{\nu\mu} = 0$$

So

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} J_\lambda F^{\lambda\nu} + \frac{F_{\mu\nu}}{2}\underbrace{\left(\partial^\mu F^{\lambda\nu} - \partial^\lambda F^{\nu\mu}\right)}_{=0}$$

$\partial^\mu F^{\lambda\nu} + \partial^\lambda F^{\mu\nu}$
sym. in μ & ν

in presence of sources the sym. stress eng. tensor for EM. field no longer divergenceless by itself. A matter term must be included in Lagrangian for matter current density

Example p^μ_{EM} not conserved $\quad p^\mu_{em} = \frac{1}{c}\int d^3x\, \Theta^{0\nu} \neq 0$

$$= \frac{1}{c}\int d^3x\, J^\rho F_\rho{}^\mu$$