So then the total cost is

$$totalcost = \frac{1}{3}n^3 + \mathcal{O}(n^2) + \frac{1}{2}n^2 + \mathcal{O}(n)$$

and you see that the solution step is a power of n cheaper than the elimination step.

8.1 Perturbation, conditioning and stability

How are solutions to matrix equations affected by (small) changes to the problem? To understand why some linear systems are difficult to solve we will investigate *conditioning*. A short example: Consider

$$A = \begin{bmatrix} 1.002 & 1 \\ 1 & 0.998 \end{bmatrix} \; ; \; b = \begin{bmatrix} 2.002 \\ 1.998 \end{bmatrix}$$

The solution is $x = (1, 1)^T$. Suppose now, the RHS is perturbed a little such that

$$b' = \begin{bmatrix} 2.0021 \\ 1.998 \end{bmatrix}.$$

This is a change of less than .5% in one component. Resolve and you will get: $x = (-23.95, 26.00)^T$. Seems incongruously large change in the solution for such a tiny change to the problem. Why? Is the large effect an artefact of the problem or the algorithm used to solve it?

First, some notational conventions and reminders from linear algebra.

Definition A vector norm on R^n is any mapping $||\cdot||$ defined on R^n with values in $[0, \infty]$ that satisfies

- $||x|| > 0$ for any vector $x \neq 0$.
- $||ax|| = |a|||x||$ for any scalar a.
- $||x + y|| \leq ||x|| + ||y||$ for any 2 vectors x and y.

Common examples: infinity norm, $||x||_\infty = \overset{max}{1 \leq i \leq n} |x_i|$ and the Euclidean 2-norm, $||x||_2 = \left(\sum_{i=1} nx_i^2 \right)^{1/2}$.

Definition Matrix norm: Let $||\cdot||$ be a given vector norm on R^n. For matrices, $A \in R^{n \times n}$

$$||A|| = \overset{max}{x \neq 0} \frac{||Ax||}{||x||}.$$

And, it follows (check!)

$$||AB|| \leq ||A||||B|| \; ; \; ||Ax|| \leq ||A||||x||.$$

The matrix infinity norm is equivalent to

$$||A||_\infty = \overset{max}{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

i.e. the maximum row sum. The matrix 2-norm is trickier

$$||A||_2 = \sqrt{\Lambda(A^T A)}$$

where $\Lambda(B)$ is the largest (absolute value) eigenvalue of the matrix B. More difficult to compute than the ∞ norm.

Example: Given

$$A = \left[\begin{matrix} 4 & -6 & 2 \\ 0 & 4 & 1 \\ 1 & 2 & 3 \end{matrix} \right)$$

check that $||A||_\infty = 12$ and $||A||_2 = 8.1659$.