Chapter 7

Tridiagonal linear systems

The solution of linear systems of equations is one of the most important areas of computational mathematics. A complete treatment is impossible here but we will discuss some of the most common problems.

Solving tridiagonal systems of equations

Recall a system of linear equations can be written

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &= f_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &= f_2 \\
\vdots \quad & \vdots \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n &= f_n
\end{aligned}
$$

where the a_{ij} and f_i are known and the x_i are unknowns. In matrix-vector form this is $Ax = f$, where A has entries a_{ij} and x and f are vectors with components x_i and f_i respectively.

A common special case is A *tridiagonal* ie there are only three diagonals in A that contain nonzero elements: the main diagonal and the superdiagonal

and subdiagonal. E.g.

$$
A = \begin{bmatrix}
a_{11} & a_{12} & 0 & \ldots & 0 \\
a_{21} & a_{22} & a_{23} & \ldots & 0 \\
0 & a_{32} & a_{33} & a_{34} & \ldots \\
\ldots & \ldots & \ldots & \ldots & a_{n-1,n} \\
0 & \ldots & 0 & a_{n,n-1} & a_{nn}
\end{bmatrix}
$$

This makes the solution of the system under certain assumptions, quite easy.

At this stage introduce some notation to simplify things: use l_i, d_i, u_i to denote the lower-diagonal, diagonal and upper-diagonal elements

$$
\begin{aligned}
l_i &= a_{i,i-1}, 2 \le i \le n \\
d_i &= a_{ii}, 1 \le i \le n \\
u_i &= a_{i,i+1}, 1 \le i \le n-1
\end{aligned}
$$

where we adopt the convention that $l_1 = 0$ and $u_n = 0$. Then the "augmented" matrix corresponding to the system is given by

and we can store the entire problem using jsut 4 vectors forl, d, u, f instead of an $n \times n$ matrix that's mostly zeroes anyway.

Recall from linear algebra methods that the standard means to solve a linear system (Gaussian elimination) is to eliminate all the components of A below the main diagonal i.e. reduce A to *triangular* form. In this case it is an easy task as we only need to eliminate a single element below the main diagonal in each column. Thus, you would multiply the first equation by l_2/d_1 and subtract this from the second equation to get

and then continue with each successive row. Note this assumes $d_1 \neq 0$ and continuing requires $d_2 - u_1(l_2/d_1) \neq 0$ etc. Under these circumstances we can reduce the system to

where $\delta_1 = d_1, \delta_2 = d_2 - u_1(l_2/d_1), \delta_3 = d_3 - u_2(l_3/\delta_2)$ and in general $\delta_k = d_k - u_{k-1}(l_k/\delta_{k-1})$ with $2 \leq k \leq n$.

Similarly,

$$g_1 = f_1, g_2 = f_2 - g_1(l_2/\delta_1), g_3 = f_3 - g_2(l_3/\delta_2)$$

so that the general form is

$$g_k = f_k - g_{k-1}(l_k/\delta_{k-1}) \quad , 2 \leq k \leq n.$$

The matrix $[T|g]$ is *row equivalent* to the original augmented matrix $[A|f]$ meaning that we can progress from one to the other using elementary row operations; thus the two augmented matrices represent systems with exactly the same solution sets. Moreover, the solution is now easy to obtain, since we can solve the last equation $\delta_n x_n = g_n$ to get $x_n = g_n/\delta_n$, and then use this value in the previous equation to get x_{n-1} and so on, to get each solution component. Again, carrying out this stage of the computation requires the assumption that each $\delta_k \neq 0$ with $1 \leq k \leq n$.

The first stage of the computation (reducing the tridiagonal matrix A to the tridiagonal one T) is generally called the elimination step and the second stage is generally called the backward solution (or *back-solve* step). A pseudocode for this process is below. Note that we don't store the entire matrix but only the three vectors needed to define the elements in the nonzero diagonals. In addition, different variable names are not used for the d_i, δ_i etc but overwrote the originals with new values. This saves storage when working with large problems.

```
/* Elimination stage */

for i=2 to n
  d(i) = d(i) - u(i-1)*l(i)/d(i-1)
  f(i) = f(i) - f(i-1)*l(i)/d(i-1)
endfor
```

```
/* Backsolve stage (bottom row is a special case) */

x(n) = f(n)/d(n)
for i=n-1 downto 1
  x(i) = ( f(i) - u(i)*x(i+1)/d(i)
```


Example

We will discuss this problem in later lectures. For now we state a common condition that is sufficient ti guarantee the tridiag solution algorithm given here will work.

Diagonal Dominance

Definition: A tridiagonal matrix is diagonally dominant if

$$d_i > |l_i| + |u_i| > 0, \ \ 1 \le i \le n$$

. Example, the matrix

$$A = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & 9 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Theorem

If the tridiagonal matrix A is diagonally dominant then the algorithm will succeed, within the limitations of rounding error.

Proof

Diagonal dominance tells us $d_1 = \delta_1 \neq 0$ so all that remains is to show that each $\delta_k = d_k - u_{k-1} l_k / \delta_{k-1} \neq 0$ for $2 \le k \le n$. Assume for the moment, $l_2 \neq 0$ then

$$\begin{aligned} \delta_2 &= d_2 - u_1 l_2 / d_1 \\ &\ge d_2 - |u_1 l_2 / d_1| \\ &\ge |u_2| + |l_2| - |l_2| \theta_1 \\ &\ge (|u_2| + |l_2|)(1 - \theta_1) \end{aligned}$$

for $\theta_1 = |u_1|/|d_1| < 1$. Therefore, $\delta_2 > 0$ since $l_2 \not\to 0$. If $l_2 = 0$ then we have $\delta_2 = d_2 - 0 = d_2 > 0$. We can repeat the argument for each index. Completes the proof.

Chapter 8

Solutions of Systems of Equations

A brief review

A vector $x \in R^n$ is an ordered n-tuple of real numbers i.e. $x = (x_1, x_2, \ldots, x_n)^T$ where the T denotes this should be considered a column vector.

A matrix, $A \in R^{m \times n}$ is a rectangular array of m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Given a square matrix, $A \in R^{n \times n}$ if there exists a second square matrix $B \in R^{n \times n}$ such that $AB = BA = I$ then we say B is the inverse of A. Note that not all sqaure matrices have an inverse. If A has an inverse it is *nonsingular* and if it does not it is *singular*.

The following theorem summarises the conditions under which a matrix is nonsingular and also connects them to the solvability of the linear systems problem.

Theorem

Given a matrix $A \in R^{n \times n}$ the following statements are equivalent:

1. A is nonsingular

2. The columns of A form and independent set of vectors
3. The rows of A form an independent set of vectors
4. The linear system $Ax = b$ has a unique solution for all vectors $b \in R^n$.
5. The homogeneous system $Ax = 0$ has only the trivial solution $x = 0$.
6. The determinant is nonzero.

Corollary

If $A \in R^{n\times n}$ is singular, then there exist infinitely many vectors $x \in R^n$, $x \neq 0$ such that $Ax = 0$.

There are a number of special classes of matrices. In particular, *tridiagonal* matrices and (later) *symmetric positive definite* matrices.

A square matrix is lower (upper) triangular if all the elements above (below) the main diagonal are zero. Thus

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

is upper tringular, while

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

is lower triangular.

Note at this stage you should review concepts of *spanning, basis, dimension and orthogonality* as we will use these ideas soon in a discussion of eignvalues and their computation.