Chapter 5

Minimums and Maximums

Extrema - the max or min point can be either:

<u>global:</u>	truly the highest or lowest function value
<u>local:</u>	highest or lowest in a finite neighborhood
	and not on the boundary of the neighborhood.

DRAWING

A,C,E	- local (not global) maxima
B,f	- local (not global) maxima
G	- global max
D	-global min

X,Y,Z 'bracket' minimum F.

Global Extrema:

In general, every difficult problem standard approaches.

- find local extrema starting from widely varying values of the independent variables - pick the most extreme of these.

- perturb a local extremum ie take a finite amplitude step away and see if you routine returns a "better" point.

5.1 Golden Section Search

The Golden section search is to find the local minimum of a curve.
It works is a similar fashion for the maximum.
This works like bracketing and bisection.
A minimum is bracketed if there's a triplet of points $a < b < c$ such that $f(b)$ is less than both $f(a)$ and $f(c)$
$\Rightarrow$ a minimum is in (a, c)
DRAWING

initial guess $(1, 2, 3)$ becomes $(4, 2, 3)$, $(4, 2, 5)$...

Algorithm

Evaluate function at some point x in the larger of (a, b) and (b, c)
if $f(x) < f(b)$
x replaces the midpoint b and b becomes an end point, (b, x, c)
if $f(x) > f(b)$
b remains the midpoint and x replaces an end point, (x, b, c)

Either way the width of the bracketing interval reduces, and the position of the minimum is better defined.
Repeat procedure until the width achieves a desired accuracy.

It can be shown that if the new test-point, x, is chosen to be a proportion

$\frac{1+\sqrt{5}}{2}$ (hence Golden Section)

along the larger sub-interval, measured from the ends, then the width of the full interval (a, c) reduces at an optimal rate.
Note The Golden section search requires no information about the derivative of f
The Golden Section is closely related to the Fibonacci Numbers

- shell spiral	- platonic solids
- plant branching	- random numbers
- flower petals	.
- pine cones	.

The Golden Section Search Algorithm

Strategy to choose new point, x, in interval given ((a, b, c):

Suppose b is a fraction ω of the way between a and c, i.e.

$$\frac{b-a}{c-a} = \omega \;\; ; \;\; \frac{c-b}{c-a} = 1 - \omega$$

Also suppose that the next trial point x is an additional fraction z beyond b,

$$\frac{x-b}{c-a} = z/$$

Then the next bracketing segment will either be of length $\omega + z$ relative to the current one or of length $1 - \omega$. We want to minimise the "worst case possibility" so choose z to make these equal. Then,

$$z = 1 - 2\omega. \tag{5.1}$$

Now, you can see that the new point is the symmetric point to b in the original interval, namely with $|b-a|$ equal to $|x-c|$. Therefore the point x lies in the larger of the two seqments.

But where? Ask where did ω come from. It would have emerged form the same strategy applied at the previous stage in the analysis. Therefore if z is chosen to be optimal then so was ω before it. This is *scale similarity* and it implies that x should be the same fraction of the way from b to c (if that's the bigger segment) as b was from a to c.

$$\begin{aligned}
\frac{x-b}{c-b} &= \frac{b-a}{c-a} \\
&= \omega \\
\frac{x-b}{c-b}\frac{c-b}{c-a} &= \omega \\
z\left(\frac{1}{1-\omega}\right) &= \omega
\end{aligned} \tag{5.2}$$

Combining eqns 5.1 and 5.2 gives

$$\omega^2 - 3\omega + 1 = 0$$

and therefore

$$\omega = \frac{3-\sqrt{5}}{2} \sim 0.38197$$

. the optimal bracketing interval has b fractional distance 0.38197 from the end and (1-0.38197)=0.61803 from the other.
These are the Golden Mean (section) from Pythagoras.

5.2 Other methods and applications

There are many other techniques for minimisation including:

- using first derivatives
- downhill simplex
- biconjugate gradient