Instructions to Candidates:

Credit will be given for the best 3 questions answered.

All questions have equal weight.

‘Formulae & tables’ are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used

In the questions $K$ denotes one of $\mathbb{R}$ or $\mathbb{C}$.

You may not start this examination until you are instructed to do so by the Invigilator.
1. (a) [6 points] If \((X, \mathcal{T})\) is a topological space define the concept of a base \(\mathcal{B}\) for \(\mathcal{T}\). What does it mean for \(\mathcal{T}\) to be second countable?

Solution:

**Definition 1.1.** If \((X, \mathcal{T})\) is a topological space, then a subfamily \(\mathcal{B} \subseteq \mathcal{T}\) of the open sets is called a base (for the open sets) of the topology if

\[ x \in U \subseteq X, \ U \text{ open} \Rightarrow \exists B \in \mathcal{B} \text{ with } x \in B \subset U \]

**Definition 1.2.** A topological space \((X, \mathcal{T})\) is called second countable if there exists a base \(\mathcal{B}\) for the topology where \(\mathcal{B}\) is a countable collection of sets.

(b) [7 points] If \(\mathcal{T}\) is the discrete topology on \(X\), show that \(\mathcal{T}\) is second countable if and only if \(X\) is countable.

Solution: If \(X\) has the discrete topology, then each singleton subset \(\{x\}\) is open and so in any base there must be a basic open set \(B\) with \(x \in B \subseteq \{x\}\). That means \(B = \{x\}\) is in the base. If the base is countable then \(X\) has to be countable. Conversely if \(X\) is countable and discrete \(\mathcal{B} = \{\{x\} : x \in X\}\) is a countable base for \(X\).

(c) [7 points] Define what it means for a topological space to be separable. Prove that second countable topological spaces are always separable. Give an example of a separable topological space which is not second countable.

Solution:

**Definition 1.3.** A topological space \((X, \mathcal{T})\) is called separable if there exists a countable subset \(S\) of \(X\) that is dense in \(X\).

**Theorem 1.4.** Second countable topological spaces are always separable.

Proof. Let \((X, \mathcal{T})\) be a second countable topological space. Let \(\mathcal{B}\) be a countable base (for the open sets of) the topology. For each nonempty \(B \in \mathcal{B}\), choose an element \(x_B \in B\). Let \(S = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}\). [In fact, if the empty set is included in \(\mathcal{B}\), we could remove the empty set from \(\mathcal{B}\) and still have a countable base.]
Now $S$ is a countable set (because the map $\mathcal{B} \setminus \{\emptyset\} \to S : B \mapsto x_B$ is a surjective map from a countable set to $S$). We claim $S$ is dense. If not $\bar{S} \neq X$ and there is $x \in X \setminus \bar{S}$. Since the complement of a closed set is open, there must be $B \in \mathcal{B}$ with $x \in B \subseteq X \setminus \bar{S}$. Now $B \neq \emptyset$ and so $x_B \in B \Rightarrow x_B \in X \setminus \bar{S}$. But $S \subseteq \bar{S} \Rightarrow X \setminus S \supseteq X \setminus \bar{S}$ and so we conclude $x_B \in X \setminus S$. But this contradicts $x_B \in S$. Hence $S$ is countable dense. Thus $X$ is second countable.

Example 1.5. We introduce a very unusual topology on the set $\mathbb{R}$ or real numbers by taking as a base all intervals of the form $(a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ (where $a < b$, $a, b \in \mathbb{R}$).

The set of real numbers with this topology is called the Sorgenfrey line.

This is separable because the rationals are dense in this topological space.

It is not second countable because if $\mathcal{B}$ is any base for the open sets in this topology, then for each $x \in \mathbb{R}$ we have $x \in [x, x + 1) = \text{an open set in this space.}$ So there must exist $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x + 1)$. Notice then that $x$ has to be the smallest element of $B_x$, thus guaranteeing that $\{B_x : x \in \mathbb{R}\}$ is an uncountable subset of $\mathcal{B}$. So there is no countable base for the topology.
2. (a) [8 points] Let \((E, \| \cdot \|)\) be a normed space. Show \(E\) is a Banach space if and only if each absolutely convergent series \(\sum_{n=1}^{\infty} x_n\) of terms \(x_n \in E\) is convergent in \(E\).

\textit{Solution:}

Proof. Assume \(E\) is complete and \(\sum_{n=1}^{\infty} \|x_n\| < \infty\). Then the partial sums of this series of positive terms

\[ S_n = \sum_{j=1}^{n} \|x_j\| \]

must satisfy the Cauchy criterion. That is for \(\varepsilon > 0\) given there is \(N\) so that \(|S_n - S_m| < \varepsilon\) holds for all \(n, m \geq N\). If we take \(n > m \geq N\), then

\[ |S_n - S_m| = \left| \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{m} \|x_j\| \right| = \sum_{j=m+1}^{n} \|x_j\| < \varepsilon. \]

Then if we consider the partial sums \(s_n = \sum_{j=1}^{n} x_j\) of the series \(\sum_{n=1}^{\infty} x_n\) we see that for \(n > m \geq N\) (same \(N\))

\[ \|s_n - s_m\| = \left\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \right\| = \left\| \sum_{j=m+1}^{n} x_j \right\| \leq \sum_{j=m+1}^{n} \|x_j\| < \varepsilon. \]

It follows from this that the sequence \((s_n)_{n=1}^{\infty}\) is Cauchy in \(E\). As \(E\) is complete, \(\lim_{n \to \infty} s_n\) exists in \(E\) and so \(\sum_{n=1}^{\infty} x_n\) converges.

For the converse, assume that all absolutely convergent series in \(E\) are convergent. Let \((u_n)_{n=1}^{\infty}\) be a Cauchy sequence in \(E\). Using the Cauchy condition with \(\varepsilon = 1/2\) we can find \(n_1 > 0\) so that

\[ n, m \geq n_1 \Rightarrow \|u_n - u_m\| < \frac{1}{2}. \]

Next we can (using the Cauchy condition with \(\varepsilon = 1/2^2\)) find \(n_2 > 1\) so that

\[ n, m \geq n_2 \Rightarrow \|u_n - u_m\| < \frac{1}{2^2}. \]

We can further assume (by increasing \(n_2\) if necessary) that \(n_2 > n_1\). Continuing in this way we can find \(n_1 < n_2 < n_3 < \cdots\) so that

\[ n, m \geq n_j \Rightarrow \|u_n - u_m\| < \frac{1}{2^j}. \]
Consider now the series $\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{\infty} (u_{n_{j+1}} - u_{n_j})$. It is absolutely convergent because

$$\sum_{j=1}^{\infty} \| x_j \| = \sum_{j=1}^{\infty} \| u_{n_{j+1}} - u_{n_j} \| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$ 

By our assumption, it is convergent. Thus its sequence of partial sums

$$s_J = \sum_{j=1}^{J} (u_{n_{j+1}} - u_{n_j}) = u_{n_{J+1}} - u_{n_1}$$

has a limit in $E$ (as $J \to \infty$). It follows that

$$\lim_{J \to \infty} u_{n_{J+1}} = u_{n_1} + \lim_{J \to \infty} (u_{n_{J+1}} - u_{n_1})$$

exists in $E$. So the Cauchy sequence $(u_n)_{n=1}^{\infty}$ has a convergent subsequence. By a Lemma $E$ is complete. 

(b) [12 points] Define the sequence space $\ell^p$ and the norm $\| \cdot \|_p$ on it for $1 \leq p < \infty$. Outline the steps required to show for $1 < p < \infty$ that $\| \cdot \|_p$ is indeed a norm and that $(\ell^p, \| \cdot \|_p)$ is a Banach space.

**Solution:**

**Definition 2.1.** For $1 \leq p < \infty$, $\ell^p$ denotes the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ which satisfy

$$\sum_{n=1}^{\infty} |a_n|^p < \infty.$$ 

It is a Banach space in the norm

$$\| (a_n) \|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}.$$ 

To prove it is a norm we rely on Young’s inequality:

**Lemma 2.2.** Suppose $1 < p < \infty$ and $q$ is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for } a, b \geq 0.$$ 

to show
Lemma 2.3 (Hölder’s inequality). Suppose $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$ this is interpreted to mean $q = \infty$ and values of $p$ and $q$ satisfying this relationship are called conjugate exponents). For $(a_n)_n \in \ell^p$ and $(b_n)_n \in \ell^q$,

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \|(a_n)_n\|_p \|(b_n)_n\|_q.$$  

(This means both that the series on the left converges and that the inequality is true.)

and

Lemma 2.4 (Minkowski’s inequality). If $x = (x_n)_n$ and $y = (y_n)_n$ are in $\ell^p$ ($1 \leq p \leq \infty$) then so is $(x_n + y_n)_n$ and

$$\|(x_n + y_n)_n\|_p \leq \|(x_n)_n\|_p + \|(y_n)_n\|_p.$$  

This is the triangle inequality for the norm $\| \cdot \|_p$ and the other properties of a norm are straightforward.

To show that it is complete (so a Banach space), we argue that every absolutely convergent series $\sum_k x_k$ in $\ell^p$ is convergent. To do that we show that the series of $n^{th}$ terms must converge since $|x_k,n| \leq \|x_k\|_p$, define a sequence $y$ by $y_n = \sum_k x_k,n$ and then show $y \in \ell^p$ and $\sum_k x_k = y$ in $\ell^p$. 
(a) [6 points] If \((X, d)\) is a metric space, define the terms nowhere dense, first category and second category (for a subset of \(X\)). State the Baire category theorem.

*Solution:*

**Definition 3.1.** A subset \(S \subset X\) of a metric space \((X, d)\) (or topological space \((X, T)\)) is called nowhere dense if the interior of its closure is empty, \((\bar{S})^o = \emptyset\).

A subset \(E \subset X\) is called of first category if it is a countable union of nowhere dense subsets, that is, the union \(E = \bigcup_{n=1}^{\infty} S_n\) of a sequence of nowhere dense sets \(S_n\) \(((\bar{S}_n)^o = \emptyset \forall n)\).

A subset \(Y \subset X\) is called of second category if it fails to be of first category.

**Theorem 3.2** (Baire Category). Let \((X, d)\) be a complete metric space which is not empty. Then the whole space \(S = X\) is of second category in itself.

(b) [7 points] Show that if \((X, d)\) is a metric space with \(X\) countably infinite and if \(X\) has no isolated points, then \((X, d)\) cannot be complete. [Hint: \(x \in X\) is called isolated if \(\{x\}\) is an open subset of \(X\)]

*Solution:*

If \(x \in X\), then \(\{x\}\) is closed and since \(x\) is not an isolated point, \(\overline{\{x\}} = \{x\}\) and so \(\{x\}\) is nowhere dense in \(X\). As \(X\) is countable \(X = \bigcup_{x \in X} \{x\}\) is a countable union of nowhere dense subsets, hence \(X\) is of first category in itself.

If \((X, d)\) was complete, since \(X\) is not empty, the Baire category theorem would contradict this. So \((X, d)\) cannot be complete.

(c) [7 points] State the open mapping theorem and use it to show that a bijective bounded linear operator between Banach spaces must have bounded inverse.

*Solution:*

**Theorem 3.3** (Open Mapping Theorem). Let \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\) be Banach spaces and \(T : E \rightarrow F\) a surjective bounded linear operator. Then there exists \(\delta > 0\) so that

\[
T(B_E) \supseteq \delta B_F
\]

where \(B_E = \{x \in E : \|x\|_E < 1\}\) and \(B_F = \{y \in F : \|y\|_F < 1\}\) are the open unit balls of \(E\) and \(F\).
Moreover, if $U \subseteq E$ is open, then $T(U)$ is open (in $F$).

**Corollary 3.4.** If $E, F$ are Banach spaces and $T : E \to F$ is a bounded linear operator that is also bijective, then $T$ is an isomorphism.

**Proof.** By the Open Mapping theorem $T$ is automatically an open map, that is $U \subseteq E$ open implies $T(U) \subseteq F$ open. But, since $T$ is a bijection the forward image $T(U)$ is the same as the inverse image $(T^{-1})^{-1}(U)$ of $U$ under the inverse map $T^{-1}$.

Thus the open mapping condition says that $T^{-1}$ is continuous. \hfill \Box
4. (a) [10 points] Let $X_n$ be first countable topological spaces for $n = 1, 2, \ldots$ and let $x = (x_1, x_2, \ldots)$ be a point in the product $\prod_{n=1}^{\infty} X_n$ and let $\mathscr{B}_n = \{B_{n,1}, B_{n,2}, \ldots\}$ be a neighbourhood base at $x_n \in X_n$ for each $n$. Assume that $B_{n,1} \supseteq B_{n,2} \supseteq \ldots \supseteq B_{n,3} \supseteq \ldots$ for each $n$.

Show that the sets of the form

$$B_{1,k} \times B_{2,k} \times \cdots \times B_{k,k} \times X_{k+1} \times X_{k+2} \times \cdots$$

make a countable neighbourhood base at $x$ (in the product topology).

**Solution:** Let $N$ be a neighbourhood of $x$. Then $x \in N^\circ$ with $N^\circ$ open in the product topology. So there is a basic open set $B$ for the product topology with $x \in B \subseteq N^\circ \subseteq N$.

Basic open sets for the product topology are finite intersections

$$B = \pi_{j_1}^{-1}(U_1) \cap \pi_{j_2}^{-1}(U_2) \cap \pi_{j_\ell}^{-1}(U_\ell)$$

where $\ell \geq 0$, $j_1, j_2, \ldots, j_\ell$ are distinct elements indices $1 \leq j_i < \infty$ and $U_i \subseteq X_{j_i}$ is open for $1 \leq i \leq \ell$. As usual, we use $\pi_j$ for the coordinate projection of the product space onto $X_j$. We can alternative write

$$B = \prod_{n=1}^{\infty} V_n$$

where

$$V_n = \begin{cases} U_i & \text{if } n = j_i \text{ for some } i, 1 \leq i \leq \ell \\ X_n & \text{otherwise.} \end{cases}$$

Alternatively we could write

$$B = V_1 \times V_2 \times \cdots \times V_r \times X_{r+1} \times X_{r+2} \times \cdots$$

where $r = \max\{j_i : 1 \leq i \leq \ell\}$.

From $x \in B$ we have $x_m \in V_m$ for $m = 1, 2, \ldots, r$.

Note that if $y = (y_1, y_2, \ldots) \in \prod_{n=1}^{\infty} X_n$ then $y \in B$ is equivalent to $y_m \in V_m$ for $1 \leq m \leq r$. 
Since \( x_m \in V_m \), \( V_m \) is open and \( \mathcal{B}_m = \{ B_{m,1}, B_{m,2}, \ldots \} \) is a neighbourhood base at \( x_m \in X_m \), there must be \( k_m \geq 1 \) with \( x_m \in B_{m,k_m} \subseteq V_m \). Since the sets in \( \mathcal{B}_m \) are in decreasing order, we then also have \( B_{m,k} \subseteq V_m \) for \( k \geq k_m \).

Let \( k_0 = \max\{ k_1, k_2, \ldots, k_r \} \). Then

\[
B_{1,k} \times B_{2,k} \times \cdots \times B_{r,k} \times X_{r+1} \times X_{r+2} \times \cdots \subseteq B
\]

for \( k > k_0 \). So, if we also ensure that \( k \geq r \) we have

\[
B_{1,k} \times B_{2,k} \times \cdots \times B_{k,k} \times X_{k+1} \times X_{k+2} \times \cdots \subseteq B_{1,k} \times B_{2,k} \times \cdots \times B_{r,k} \times X_{r+1} \times X_{r+2} \times \cdots \subseteq B.
\]

So we have the neighbourhood base property, but finally we need to check that the sets

\[
B_{1,k} \times B_{2,k} \times \cdots \times B_{m,k} \times X_{k+1} \times X_{k+2} \times \cdots
\]

are neighbourhoods of \( x \). That is quite easy since \( x_m \in B_{m,k}^\circ \) (for all \( m \)) and

\[
B_{1,k}^\circ \times B_{2,k}^\circ \times \cdots \times B_{m,k}^\circ \times X_{k+1} \times X_{k+2} \times \cdots
\]

is an open set in the interior containing \( x \).

(b) [10 points] Let \( x_n = (x_{n,j})_{j=1}^\infty \) denote the sequence with \( j \)th term

\[
x_{n,j} = \begin{cases} 
\frac{1}{n} & \text{if } j = n \\
0 & \text{for } j \neq n
\end{cases}
\]

Show that \( \sum_{n=1}^\infty x_n \) is convergent in \( \ell^p \) for \( 1 < p < \infty \) but fails to be absolutely convergent.

**Solution:** If we write out \( x_n \) in longhand, we find

\[
x_n = (0, 0, \ldots, 0, \frac{1}{n}, 0, \ldots)
\]

(or we could say that the sequence has all zero terms apart from the \( n \)th term, which is \( 1/n \)). So if we compute for \( 1 \leq p < \infty \), we find

\[
\|x_n\|_p = \left( \sum_{j=1}^\infty |x_{n,j}|^p \right)^{1/p} = \left( 0^p + 0^p + \cdots + 0^p + \left( \frac{1}{n} \right)^p + 0^p + \cdots \right)^{1/p}
\]
and so we get \[ \|x_n\|_p = (1/n^p)^{1/p} = 1/n. \]

So
\[
\sum_{n=1}^{\infty} \|x_n\|_p = \sum_{n=1}^{\infty} \frac{1}{n}
\]
and this is infinite (harmonic series does not converge).

So the series \( \sum_{n=1}^{\infty} x_n \) is not absolutely convergent in \( \ell^p \) (no matter what \( p \) we choose).

The partial sums \( S_n = x_1 + x_2 + \cdots + x_n \) work out as
\[
x_1 = (1, 0, 0, \ldots)
x_2 = (0, \frac{1}{2}, 0, 0, \ldots)
\vdots
x_n = (0, 0, \ldots, 0, \frac{1}{n}, 0, \ldots)
\]
\[
S_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)
\]

It seems reasonable to guess that the limit of these \( S_n \) must be the sequence
\[
S = (1, \frac{1}{2}, \frac{1}{3}, \ldots) = \left( \frac{1}{j} \right)_{j=1}^{\infty}
\]
and we can see if that is so by looking at \( \lim_{n \to \infty} \|S_n - S\|_p \). We have
\[
S - S_n = (0, 0, \ldots, 0, -\frac{1}{n+1}, -\frac{1}{n+2}, \ldots),
\]
and for \( 1 < p < \infty \)
\[
\|S_n - S\|_p = \left( 0^p + 0^p + \cdots 0^p + \left| -\frac{1}{n+1} \right|^p + \left| -\frac{1}{n+2} \right|^p + \cdots \right)^{1/p}
\]
\[
= \left( \sum_{j=n+1}^{\infty} \frac{1}{j^p} \right)^{1/p}.
\]

For \( 1 < p < \infty \), the series \( \sum_{j=1}^{\infty} \frac{1}{j^p} \) converges, so that the tail sums \( \sum_{j=n+1}^{\infty} \frac{1}{j^p} \to 0 \) as \( n \to \infty \).