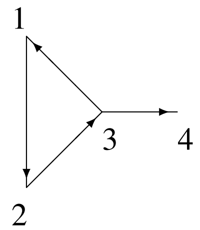
1S11 (Timoney) Tutorial sheet 10
[December 4 – 7, 2012]

Name: Solutions

1. Write a vertex matrix for this directed graph



Solution: In row i we put a 1 in column j when there is a link $i \to j$. Otherwise 0. We get the 4×4 matrix

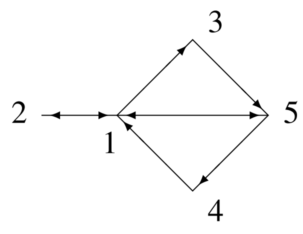
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Draw a graph for this vertex matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and then use matrix multiplication to find all the possible 2-hop directed paths on the graph.

Solution:



$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

This shows the the 2-hop paths are $1 \to 5$, $1 \to 4$, $2 \to 3$, $2 \to 5$, $3 \to 1$, $3 \to 4$, $4 \to 2$, $4 \to 3$, $4 \to 5$ $5 \to 1$ $5 \to 2$ $5 \to 3$ and $5 \to 3$, in addition to the possibility for round trips $1 \to 1$ and $2 \to 2$.

3. (a) Is there a matrix that is both strictly upper and strictly lower triangular? (Explain why not or else give an example of one.)

   Solution: The zero matrix of any size, such as the 2×2 matrix

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   is both strictly upper triangular (all nonzero entries definitely above the diagonal — in fact there are none) and strictly lower triangular.

   [In fact no other matrix will do, except the $n \times n$ [square] zero matrix fo any n.]

   (b) If A and B are matrices with

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

   what is $\mathrm{trace}(BA)$?

   Solution: We know $\mathrm{trace}(BA) = \mathrm{trace}(AB)$ and so

$$\mathrm{trace}(BA) = \mathrm{trace}(AB) = 1 - 3 + 2 = 0$$

4. For

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -8 & 3 & 1 & 0 \end{bmatrix}$$

   find B^2, $(B^t)^2$, B^3, $(B^t)^3$ and B^4. [Hint: use the rule about products of transposes.]

   Solution:

$$B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 \end{bmatrix}$$

   Using $(AB)^t = B^t A^t$ (or $B^t A^t = (AB)^t$) we have

$$(B^t)^2 = (B^t)(B^t) = (BB)^t = (B^2)^t = \begin{bmatrix} 0 & 0 & 3 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next

$$B^3 = BB^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -8 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -7 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$(B^t)^3 = (B^3)^t = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally

$$B^4 = BB^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -8 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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