Chapter 1. Vectors

This material is in Chapter 3 of Anton & Rorres. We will discuss to the topics earlier in the book later.

Vectors have quite a few practical uses. They are used directly and can be thought of in gemetrical terms but they can also be manipulated algebraicly, rather like the way we can use coordinate geometry to compute with genetrical shapes. Vectors also provide a language which can used in more general contexts, where the pictures no longer apply, for instance in manipulating data or extracting certain best fit patterns for data (like find a line that best fits a bunch of points in the plane, which might represent the data from an experiment).

1.1 Scalar and vector quantities

Some quantities have numerical values (like mass, volume, temperature) while others also have a direction associated with them.

Examples of quantities with a direction include:

(i) Wind velocity — which means both the speed (a numerical value) and the direction (like from the West, blowing towards the East);

(ii) Force — will have a strength and a direction in which it acts;

(iii) displacement — for example a displacement (or movement) of 1 km North Eastwards.

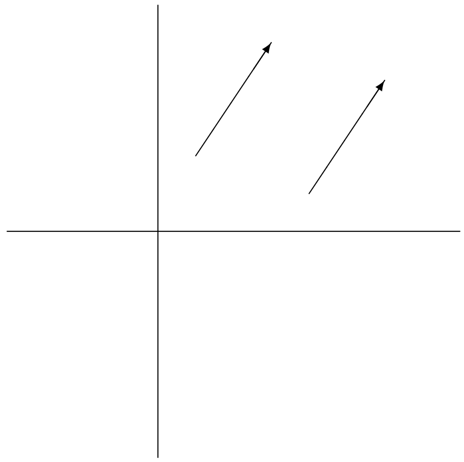
All these 3 are examples of *vector* quantities. They have a magnitude and a direction but we don't consider them as fixed at any place. In this contect we may use the term *scalar* for a numerical quantity (with no direction), especially when we want to emphasise that it is not a vector.

1.2 Vectors as arrows

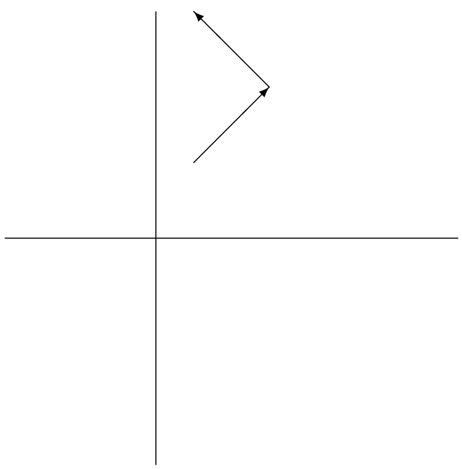
Graphically, or geometrically, we think of a vector as pictured by an arrow where the length of the arrow represents the magnitude and the direction the arrow points in gives us the direction.

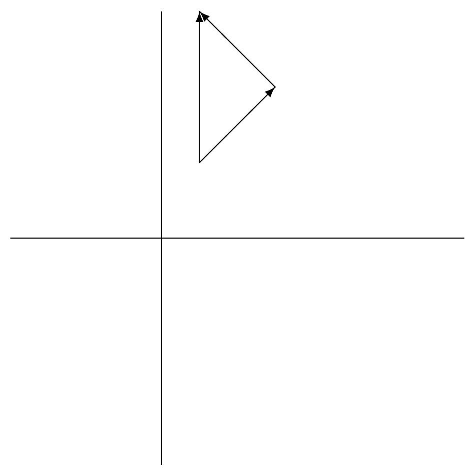
These vectors can be in 2 dimensions (a plane) or 3 dimensions (space). Let's think about 2 dimensions first. We then regard vectors as arrows in a plane.

Here is a picture of two parallel arrows, with the same length and direction. So they are pictures of the same vector (because we don't consider vectors as fixed at any position).



We can perhaps 'discover' how to add vectors if we think about displacements as an example. If you are displaced 1 unit NorthEast and then 1 unit NorthWest, where to you end up?



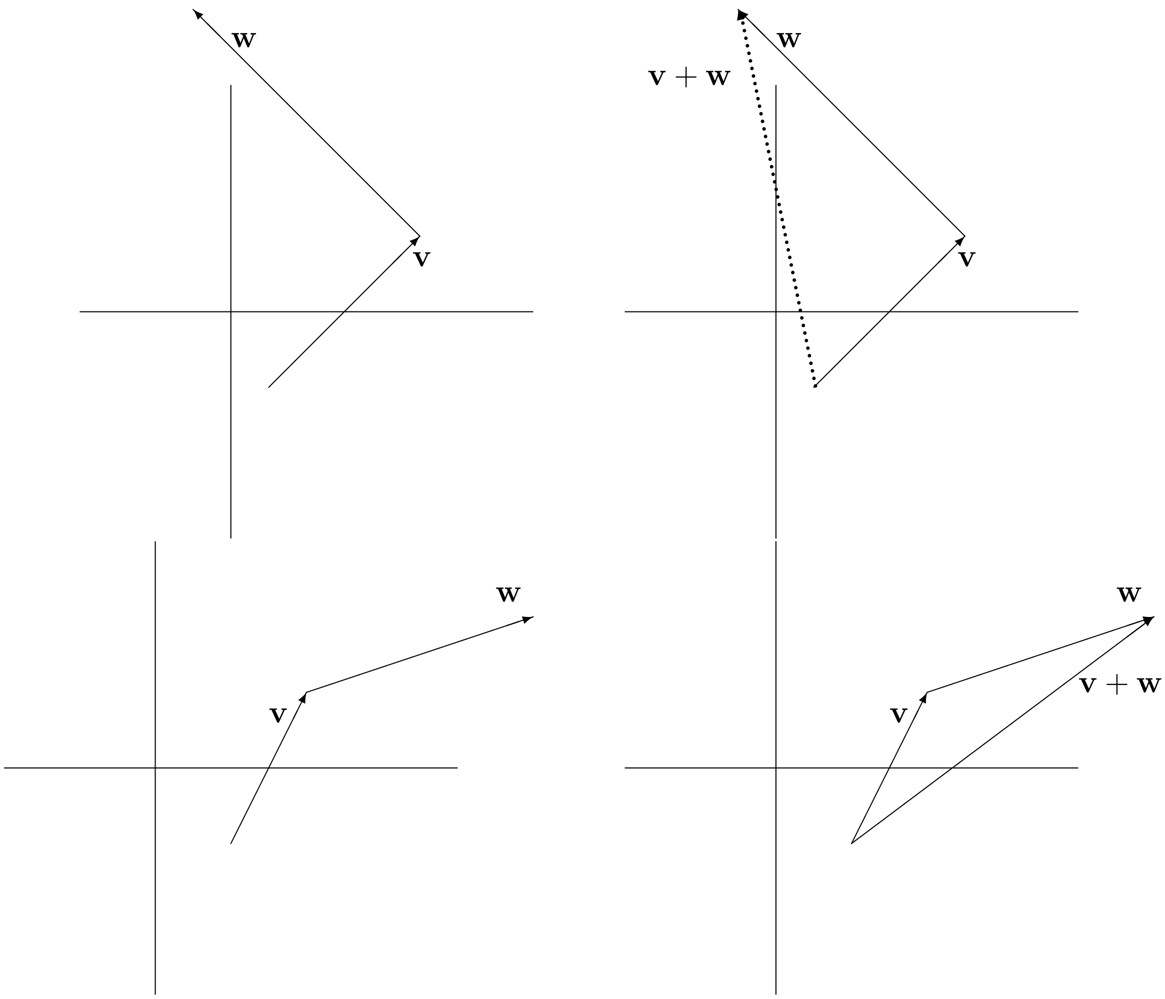


Because the two displacements make a right angle with each other, we can easily work out a precise answer. You would end up due North of where you started, and a distance $\sqrt{2}$ away from the start.

This is an example of what we might refer to as the triangle rule for adding vectors:

Rule: To add two vectors $\mathbf{v}$ and $\mathbf{w}$, place the arrows representing them 'nose to tail' and take the third side of the triangle for the vector $\mathbf{v} + \mathbf{w}$.

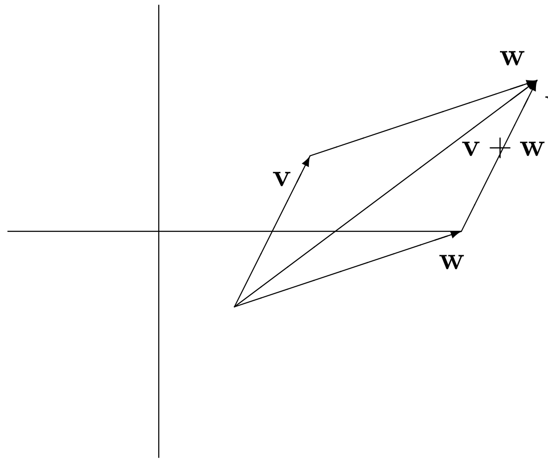
Here are pictorial examples.



Note that when we write things down by hand we use $\vec{v}$ to indicate a vector called v, but in print it is also common to use bold face letters $\mathbf{v}$ for vectors. (Some people use underlines, so $\underline{v}$ rather than $\vec{v}$, when writing by hand.)

Fact: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ always.

You can convince yourself of this fact by looking at a diagram like



The triangles you would make for $\mathbf{v}+\mathbf{w}$ and for $\mathbf{w}+\mathbf{v}$ will form a parallelogram if you start them at the same point. And the sum of the two vectors appears as the diagonal of a parallelogram (the diagonal from the starting point).

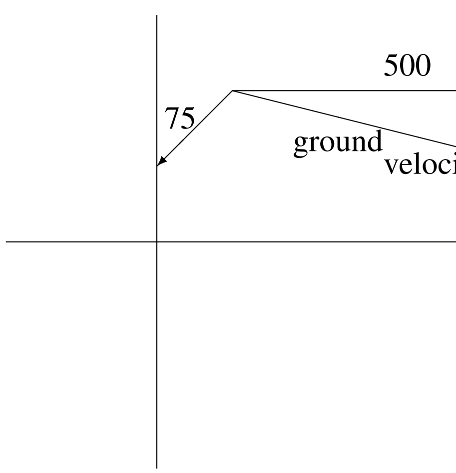
This shows an alternative rule for addition (the parallelogram rule):

Rule: To add two vectors $\mathbf{v}$ and $\mathbf{w}$, place the arrows representing them starting at the same point, complete a parallelogram using the two vectors as two of the sides, and then the diagonal of the parallelogram (the one with the same starting point as the two vectors) will be $\mathbf{v}+\mathbf{w}$.

Experimental fact: If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are acting on the same object, their combined effect is the same as that of one force given by the vector $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$. (The sum of two force vectors is sometimes called the resultant force. To state the experiment more carefully we should say that the forces act on an object with no extent, a particle.)

Another fact: Velocities also add by the same rule for vector addition.

A example of this can be provided by an aeroplane. Say the plane is moving with an airspeed of 500 km/h heading due East, but the air is moving (wind blowing) towards the SouthWest at 75 km/h. Then the ground velocity of the plane is got by adding the two vectors



The resultant ground velocity (speed and direction) will be somewhat more to the SouthEast than due East and have a magnitude smaller than 500.

You can also think about a boat traveling in moving water and come to similar conclusions.

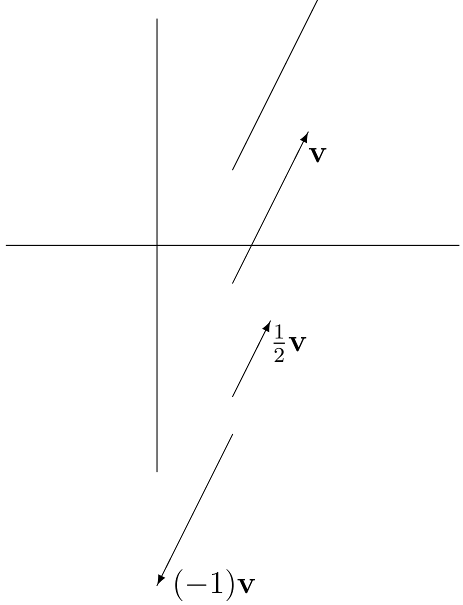
Although it looks different, this is not so different from the conclusions we reached for displacements. Think of the displacement in 1 hour (or 1 minute) of the plane and the air. In 1 hour the plane will have travelled 500 km East through the air, but the air will have moved 75 km to the SouthWest. (An hour is a long time and things might change in that time. So doing things over a shorter time interval would be more realistic. In one minute the plane would move $500/60$ km and the air $75/60$ km. So every displacement would be scaled down in magnitude by a factor of 60, but the directions will be the same.)

1.3 Multiples of vectors

We've seen how to add vectors and now we describe how to multiply vectors by scalars.

If $\mathbf{v}$ is a vector then $2\mathbf{v}$ is the vector with the same direction as $\mathbf{v}$ but twice the magnitude (or twice the length). The same rule works for $k\mathbf{v}$ for any positive scalar k. The vector $k\mathbf{v}$ has the same direction as $\mathbf{v}$ but k times the length (for $k > 0$).

For negative k we have a variation on this. $(-1)\mathbf{v}$ is the vector with the same length as $\mathbf{v}$ and exactly the opposite direction. Here is a picture showing some different multiples of a vector $\mathbf{v}$.



For (say) $(-5)\mathbf{v}$ we take 5 times $(-1)\mathbf{v}$. Or another way to describe it is that if $k < 0$, then $k\mathbf{v}$ is the vector with exactly opposite direction to $\mathbf{v}$ and $|k|$ times the length of $\mathbf{v}$.

If you think about this a little, you will notice that the case of $k = 0$ has not been covered. $0\mathbf{v}$ still has to be defined. We define it to have length zero, but agree that the direction of a vector with zero magnitude is immaterial.

What that means is that we allow for one *zero vector*, a vector with length 0 and indeterminate direction. We can denote it by $\mathbf{0}$ or $\vec{0}$.

Then we define $0\mathbf{v} = \mathbf{0}$.

You might not be convinced that have a zero vector is a sensible thing. But if you think why we have a number zero. We often need it (for example to indicate a zero balance of a bank account) and similarly it is possible to add some vectors to end up with them all canceling. A simple situation is of two equal and opposite forces acting. The resultant force (which will be no force) will be represented by the zero vector. Or if you are swimming upstream at 5 km/h and the water is flowing down at 5 km/hr, then your velocity will be $\mathbf{0}$.

1.4 Rules for vector arithmetic

The notations $\mathbf{v} + \mathbf{w}$ and $k\mathbf{v}$ are so suggestive of all the ordinary rules of algebra that you might be willing to believe they hold true. Indeed they do, but really we should check them out. Here are examples

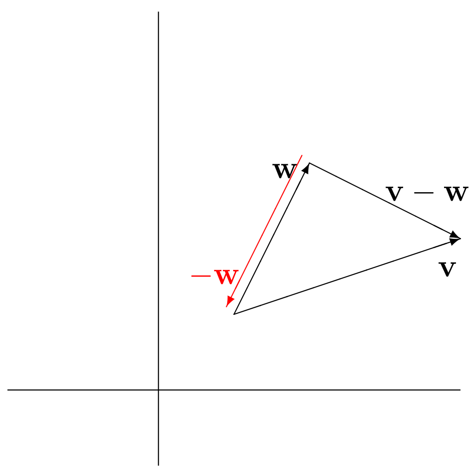
$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) \\ 3(\mathbf{u} + \mathbf{v}) &= 3\mathbf{u} + 3\mathbf{v} \end{aligned}$$

(for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$).

We use $\mathbf{v} - \mathbf{w}$ for $\mathbf{v} + (-1)\mathbf{w}$.

A very useful little fact is that if you draw the vectors $\mathbf{v}$ and $\mathbf{w}$ as arrows starting at the same point, then the vector $\mathbf{v} - \mathbf{w}$ is represented by the arrow from the end of $\mathbf{w}$ to the end of $\mathbf{v}$.

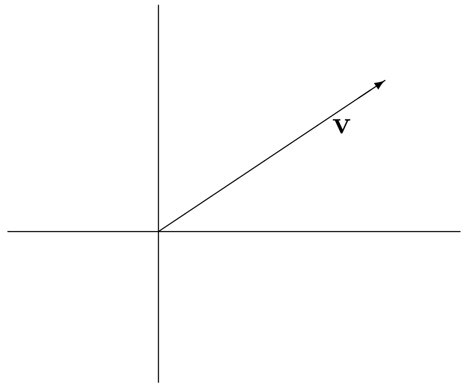
To see why that is true, draw a little diagram. $-\mathbf{w}$ is a vector in the exact opposite direction as $\mathbf{w}$. You can draw it on top of the arrow you drew for $\mathbf{w}$ if you just put the arrow at the other end. Then you have $-\mathbf{w}$ and $\mathbf{v}$ nose to tail and you can add them by the triangle rule.



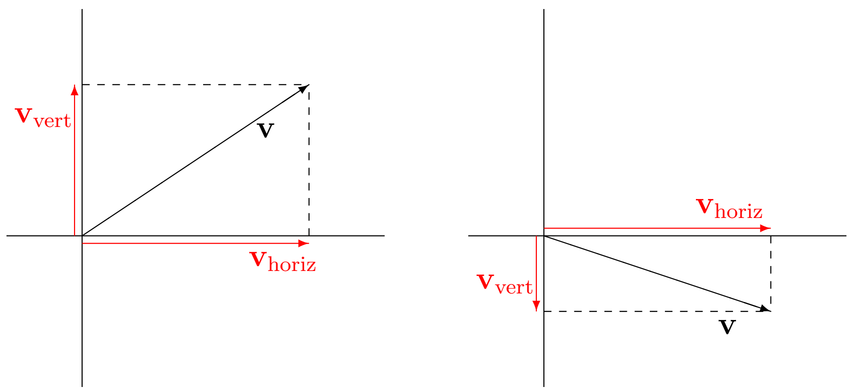
1.5 Components of vectors in the plane

Suppose we are given a vector $\mathbf{v}$ in the plane. When we say 'the plane' we usually suppose that the plane already has two fixed axes in it, an x-axis and a perpendicular y-axis. The axes each have a positive direction and we have a scale (notion of unit length) on each axis.

We can choose (if we like) to draw $\mathbf{v}$ as a vector starting at the origin $(0, 0)$ in the plane.



We can always write $\mathbf{v}$ as a sum $\mathbf{v} = \mathbf{v}_{\text{horiz}} + \mathbf{v}_{\text{vert}}$ of a horizontal vector $\mathbf{v}_{\text{horiz}}$ and a vertical vector $\mathbf{v}_{\text{vert}}$

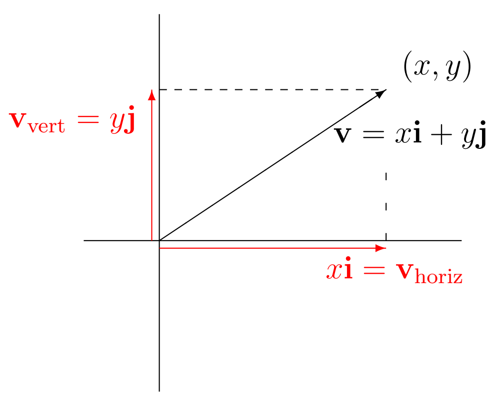


We also introduce two *standard basis vectors* $\mathbf{i}$ and $\mathbf{j}$. The vector $\mathbf{i}$ is the *unit vector* (which is the term we use for a vector of length 1) in the direction of the positive x-axis. $\mathbf{j}$ is the unit vector in the direction of the positive y-axis.

We can write $\mathbf{v}_{\text{horiz}}$ as a multiple of $\mathbf{i}$. We can even do this if $\mathbf{v}_{\text{horiz}}$ is pointing to the left, by the use of negative multiples. We multiply $\mathbf{i}$ by the length of $\mathbf{v}_{\text{horiz}}$ to get $\mathbf{v}_{\text{horiz}}$ (if $\mathbf{v}_{\text{horiz}}$ points to the right, and by minus the length if $\mathbf{v}_{\text{horiz}}$ points to the left).

Similarly we can write $\mathbf{v}_{\text{vert}}$ as a multiple of $\mathbf{j}$. In fact it works that if (x, y) are the coordinates of the end point of $\mathbf{v}$ (when we draw $\mathbf{v}$ starting at $(0, 0)$ as we have done already), then

$$\mathbf{v}_{\text{horiz}} = x\mathbf{i}, \quad \mathbf{v}_{\text{vert}} = y\mathbf{j}$$



We end up with

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j}$$

To explain this again in terms that are perhaps easier to remember, we introduce the notion of the *position vector* of a point (x, y) in the plane. Geometrically, the position vector of a point P is the vector represented by the arrow from the origin $(0, 0)$ to P. (If we denote the origin by the letter O, then people sometimes write $\vec{OP}$ for this arrow from O to P.) So we can see that every point P in the plane gives rise to a vector, the position vector of P. Also, looking at the pictures above we can see that if the coordinates of P are (x, y) then the position vector of P is the vector $x\mathbf{i} + y\mathbf{j}$.

On the other hand, if we start with any vector $\mathbf{v}$, we can always draw it as an arrow starting at the origin $(0, 0)$. But then it will be the position vector of the point where it ends.

In this way we can go back and forth from points with coordinates (x, y) to the vector $x\mathbf{i}+y\mathbf{j}$.

If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, we call x and y the *components* of $\mathbf{v}$ in the directions of the two axes.

A little later on, we will get sufficiently used to the idea of converting points (x, y) into their position vector $x\mathbf{i} + y\mathbf{j}$, and going backwards from the vector to the point that it will become very tempting to consider the point and the vector as almost the same thing. But, for now we will keep the distinction between points and vectors fairly clear.

You can use components and the rules of vector algebra we mentioned above (in 1.4) to calculate things quite easily. For instance if $\mathbf{v} = 4\mathbf{i} - 7\mathbf{j}$, then

$$5\mathbf{v} = 5(4\mathbf{i} - 7\mathbf{j}) = 20\mathbf{i} - 35\mathbf{j}$$

If $\mathbf{w} = -3\mathbf{i} + \mathbf{j}$, then

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (4\mathbf{i} - 7\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\ &= (4 - 3)\mathbf{i} + (-7 + 1)\mathbf{j} \\ &= \mathbf{i} - 6\mathbf{j} \end{aligned}$$

1.6 Length of a vector

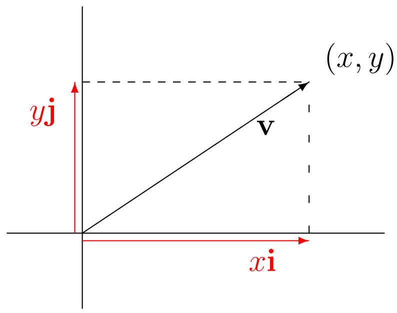
We use the notation $\|\mathbf{v}\|$ to stand for the length (or magnitude) of a vector $\mathbf{v}$. From the way we defined multiples $k\mathbf{v}$ of a vector $\mathbf{v}$ (by scalars k) we see that we have

$$\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$$

always.

When we work in components, we want to express $\|\mathbf{v}\|$ in terms of components. In 2 dimensions, when we look at $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ we can use Pythagoras' theorem to show

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$



To check this look at the right-angled triangle with the vector along the hypoteneuse, horizontal side of length $|x|$ and vertical side of length $|y|$.

1.7 Dot products

It is sometimes easier to remember which component belongs to which vector if we use subscripted (not bold face) letters for the components of a vector, like

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}, \qquad \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j},$$

We have already dealt with addition of vectors (adding one vector to another) and multiplication of a vector by a scalar. But we have not talked about multiplication of one vector by another. Part of the reason for this is that there is something a bit unsatisfying about the ways we multiply vectors together.

We define the *dot product* (also know as the *scalar product*) of two vectors $\mathbf{v}$ and $\mathbf{w}$ now. It is denoted $\mathbf{v} \cdot \mathbf{w}$ and it is a scalar or numerical quantity (**not** another vector). In terms of the components of $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$ we define

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

to be the number you get by multiplying the first component of $\mathbf{v}$ by the first component of $\mathbf{w}$, the second by the second, the third by the third **and then adding** these numbers together.

So if (say) $\mathbf{v} = 11\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$, we get

$$\mathbf{v} \cdot \mathbf{w} = 11(3) + (-2)(4) = 25$$

It is really useful to keep in mind that the dot product has a scalar value. Obviously then, if you get a vector answer, it could not possibly be right.

The dot product satisfies some nice algebraic rules, which you might be inclined to take for granted. In principle they should be verified, but the verification is not very difficult and we will not do it. Here are the basic rules, satisfied by any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any scalar k

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$
$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(iii) $(k\mathbf{v})\cdot\mathbf{w} = k(\mathbf{v}\cdot\mathbf{w}) = \mathbf{v}\cdot(k\mathbf{w})$

To 'prove' that these are valid rules, valid for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any scalar k, what one does is to write out each vector in components, then write out all the dot products in terms of those components. If you expand out everything you see that you get the same results (for both sides of each of the claimed equations).

(iv) Another property that is very useful is

$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$$

This is easy to check also. If you write $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ and work out

$$\mathbf{v}\cdot\mathbf{v} = v_1v_1+v_2v_2+v_3v_3 = v_1^2+v_2^2+v_3^2$$

you see it agrees with $\|\mathbf{v}\|^2$.

(v) We can use this and the cosine rule for triangles to show a geometrical way of describing the dot product. It is

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

where θ stands for the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$. That means we draw $\mathbf{v}$ and $\mathbf{w}$ as arrows staring at the same point and measure the angle they make with one another, the smallest positive angle. It will be a value between 0 (if the vectors have the same direction) and π (if they are pointing in exactly opposite directions).

It is good to use **radians** for angles, rather than degrees.[1]

Example. Find the angle between the vectors $\mathbf{v} = 2\mathbf{i}-3\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i}+2\mathbf{j}$

We calculate everything in the formula $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ except $\cos\theta$. We get

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{2^2+(-3)^2}\\
&= \sqrt{13}\\
\|\mathbf{w}\| &= \sqrt{5^2+2^2}\\
&= \sqrt{29}\\
\mathbf{v}\cdot\mathbf{w} &= (2)(5)+(-3)(2)\\
&= 10-6\\
&= 4\\
0 &= \sqrt{13}\sqrt{29}\cos\theta\\
\cos\theta &= 4/(\sqrt{13}\sqrt{29})
\end{aligned}$$

[1] All of calculus for trig functions depends on the fact that we use radians. If we used degrees the derivative of $\sin x$ would not be $\cos x$, but would have a factor $\pi/180$ in the formula. In physical terms, radians are dimensionless, but degrees are not.

and so $\theta = 1.3633$ (radians) in this case.

Orthogonal vectors. We can see from the formula that two vectors $\mathbf{v}$ and $\mathbf{w}$ will be perpendicular if $\mathbf{v} \cdot \mathbf{w} = 0$.

Well, to be more accurate, we can conclude this only when $\mathbf{v}$ and $\mathbf{w}$ are both nonzero vectors. If any one of $\mathbf{v}$ and/or $\mathbf{w}$ is the zero vector then they will have length zero. So we can't divide the equation $0 = \|\mathbf{v}\|\|\mathbf{v}\| \cos\theta$ across by the lengths to find $\cos\theta$. In fact, the problem is that we can't talk about the angle between the zero vector and any other vector because we agreed that the zero vector should have no particular direction. To get around this we make the *convention* that the zero vector is to be considered perpendicular (or orthogonal is another word for the same idea) to every other vector.

With this convention in place we can say that two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal exactly when $\mathbf{v} \cdot \mathbf{v} = 0$.

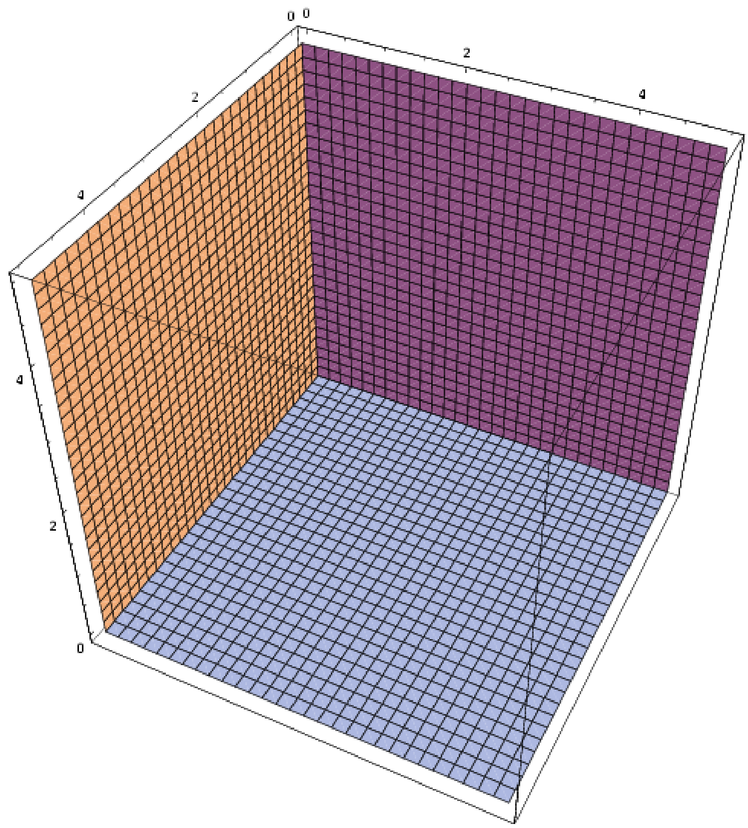
1.8 Vectors in space

The same rules work (from §1.2 and §1.3) to define $\mathbf{v} + \mathbf{w}$ and $k\mathbf{v}$ if we allow $\mathbf{v}$ and $\mathbf{w}$ to be vectors in space. The parallelogram (or triangle) rule for $\mathbf{v} + \mathbf{w}$ will work but we have to think of the vectors in 3 dimensional space, while the parallelogram (or triangle) will still be flat. The parallelogram (or triangle) will lie in some plane inside space.

1.9 Coordinates and vector components in space

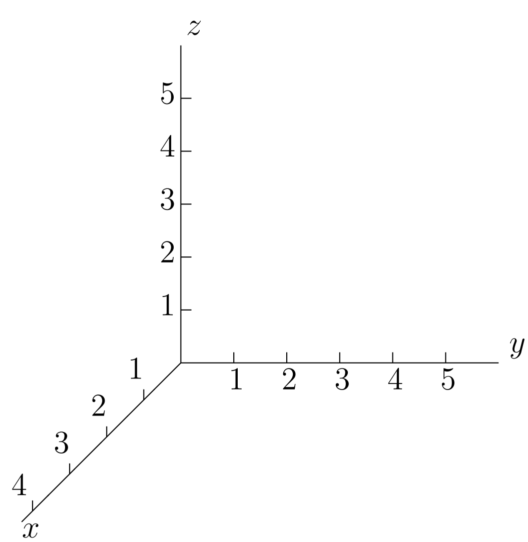
We are used to using two coordinates (x, y) to locate points in a plane (with reference to two fixed perpendicular axes). For points in space we need 3 coordinates and 3 axes, and the need to think spatially makes it a little harder to explain and to grasp. It does not help that pictures on a page can sometimes not convey the intended 3-dimensional effect until you look at them correctly.

One way to think about how coordinates work in space is to imagine 3 perpendicular axes meeting at the corner of a room (an ordinary boxy shaped room). Here is a picture of the kind of corner I have in mind



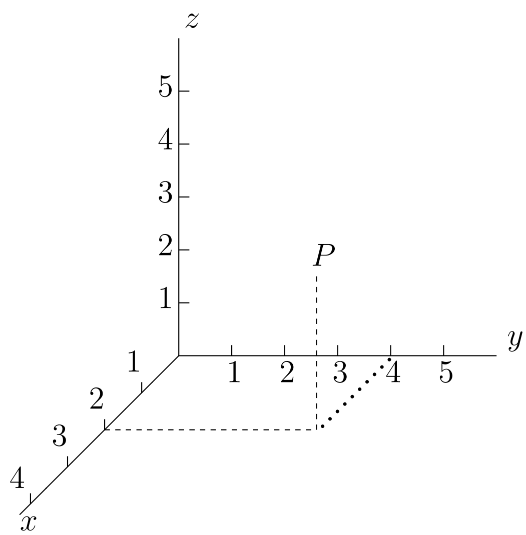
Look at the blue bit as the floor and the brown and purple as walls. So we are looking down and to the left into this corner.

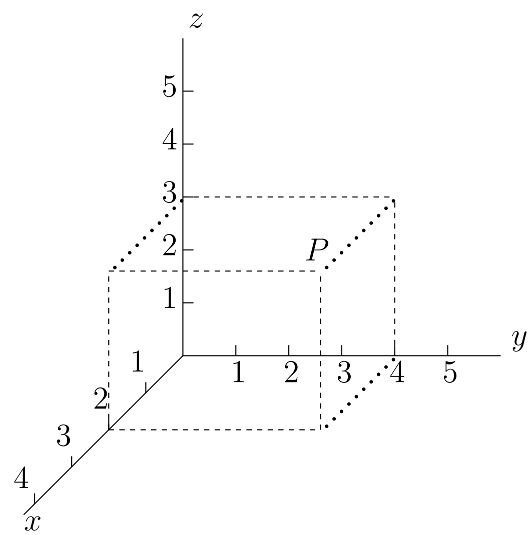
We think of 3 axes meeting at the corner, one along the floor to the left which we will call the x-axis, one along the floor at the back called the y-axis and the third vertical (where the walls meet) and we call that the z-axis. We are more often going to draw just the 3 axes like this



You can see we need to imagine the x-axis as coming out of the picture towards us. And we have a scale on each axis.

To describe where a point is with our room, we do it in two steps. First we describe a point on the floor directly below our point. For this we need just two coordinates (x, y). Then we use a third coordinate z to give a height (or altitude) of our point above the floor. Here is an example where $x = 2$, $y = 4$ and $z = 3$.





In the second picture (to the right) you can see a box outlined. The box has its corner at the corner of our "room" and the point P we are describing is at the corner of the box farthest away from where our origin is (which is at the corner of the room).

One additional thing to realise is that coordinates can be negative as well as positive. When $z = 0$ we are at height 0, or on the floor, or on the horizontal x-y plane. When $z < 0$ we mean a depth below the floor when we give a negative height. So maybe that is a point in the room below if you are in an upper floor of some building. When $x < 0$ we are at a point behind the back wall, and when $y < 0$ we are at a point to the left of the wall to our left (on the other side of the x-z plane that contains the x and z axes).

Once you digest this way of thinking, you see that we can describe points in space by 3 coordinates (x, y, z). Given a point you can find coordinates for it, and given coordinates you can locate the point precisely. (All this is supposing you know where your axes are.)

Now, moving to vectors in space, we can do what we did with vectors in the plane. We introduce 3 standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, which are the unit vectors in the positive directions along the x- y- and z-axes.

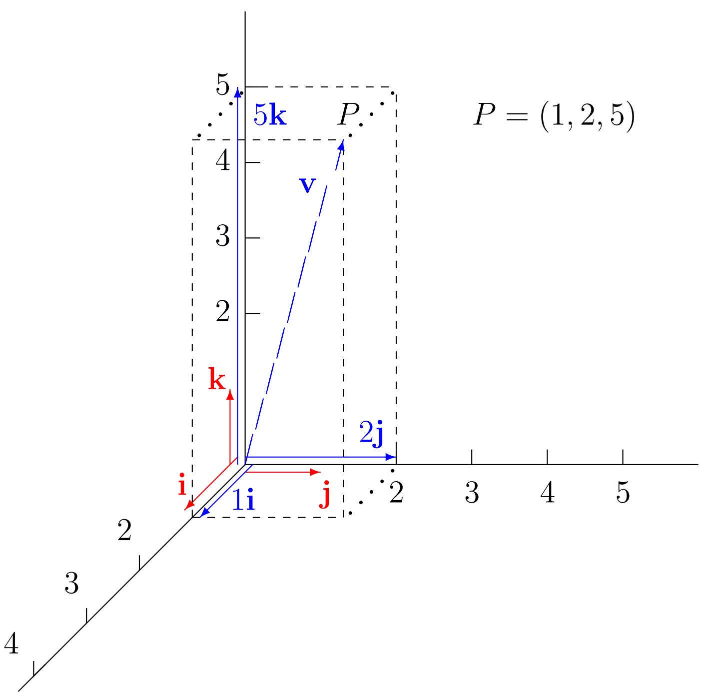
Every vector $\mathbf{v}$ is space can be drawn as an arrow starting from the origin $(0, 0, 0)$. We can write $\mathbf{v}$ as a sum of a horizontal and a vertical vector. Then we can write the horizontal one (in the horizontal plane) as a sum of two perpendicular (still horizontal) vectors parallel to the x- and y-axes. The end result is that we can write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where (x, y, z) is the point where $\mathbf{v}$ ends.

The following picture illustrates this when $\mathbf{v}$ ends at $(1, 2, 5)$ and so

$$\mathbf{v} = 1\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$



In this way we can, as we did in the plane, go from a point $P = (x, y, z)$ to its position vector

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and from a vector back to the point for which it is the position vector.

We can also do computations of (say) $7\mathbf{v} + 6\mathbf{w}$ with components. It is really no harder than in 2 dimensions although there is one extra component to each vector, and that adds to the amount of arithmetic.

1.10 Length of a vector (in space)

For a vector in 3 dimensions, $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, you need to use Pythagoras' theorem twice. We know from the one use of the theorem (in the x-y plane) that

$$\|x\mathbf{i} + y\mathbf{j}\| = \sqrt{x^2 + y^2}$$

and the vector $\mathbf{v}$ makes a right angled triangle with that vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ and a vertical (perpendicular) vector $z\mathbf{k}$. So

$$\|\mathbf{v}\|^2 = \|x\mathbf{i} + y\mathbf{j}\|^2 + |z|^2 = x^2 + y^2 + z^2$$

In summary then, for a vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we have

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

(a very similar formula to the familiar one in two dimensions, with just the extra component z).

For example, if $\mathbf{v} = -\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, we can compute

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 4^2 + (-7)^2} = \sqrt{(-1)^2 + 4^2 + (-7)^2} = \sqrt{66}.$$

1.11 Distance formula in space

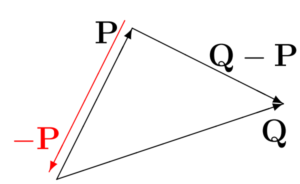
You should recall having learned the distance formula for points in the plane:

$$\mathrm{distance}((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

There is a similar formula in space.

Here is a rationale for it based on vector ideas.

If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are two points in space, we think about their position vectors (represented by the arrows from the origin to P and to Q). Recall that we saw before that $\mathbf{Q} - \mathbf{P}$ is the vector represented by the arrow from P to Q (which we sometimes write $\vec{PQ}$).



But we can see then that the length $\|\mathbf{Q} - \mathbf{P}\|$ of the vector $\mathbf{Q} - \mathbf{P}$ must be exactly the distance from P to Q. So we get

$$\begin{aligned}\mathrm{dist}(P, Q) &= \|\mathbf{Q} - \mathbf{P}\| \\ &= \|(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\| \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

So the distance formula looks like an obvious extension of the one in 2 dimensions. You just take account of the third coordinate.

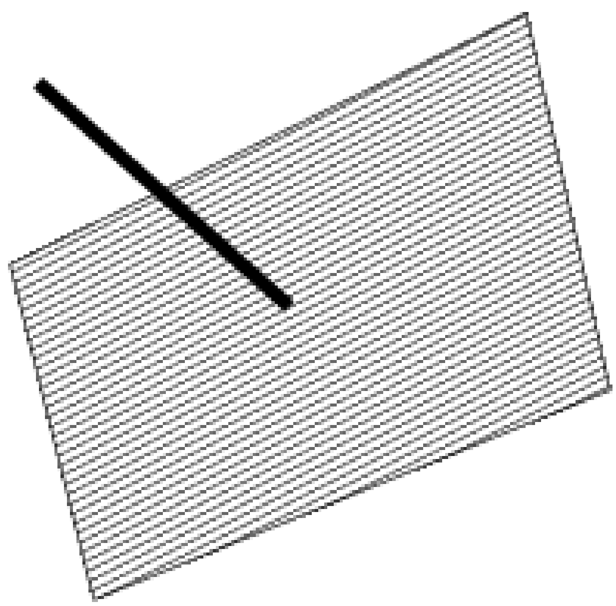
To give an example with numerical values

$$\mathrm{dist}((4, 5, 6), (7, 2, -1)) = \sqrt{(7 - 4)^2 + (2 - 5)^2 + (-1 - 6)^2} = \sqrt{67}$$

1.12 Equations of planes (through the origin)

We consider first the question of how we might describe the orientation of a plane in space. For lines in the plane we use the slope usually (though there are vertical lines $x = a$ that have no slope). We could perhaps think about the slope of a plane as the slope of the steepest line up the plane, but this will not tell us the orientation of our plane in space. We can move the plane around (say rotating it around a vertical direction), keeping the same largest slope, but changing the way the plane is oriented in space.

If you think about it for a moment, you will see that the method of picking a vector perpendicular to the plane (which we sometimes call a *normal vector* to the plane) works well. You could maybe construct a little model, like plasterers use, or a flat board with a stick handle perpendicular to the board (attached by glue or a screw to one side of the board). If you want to move the board around parallel to itself you have to keep the stick handle pointing in a constant direction. (To be more precise, you could also turn the handle around the other way, so that it will reverse direction completely, and still have the board in the same plane.)



You should see that picking a vector perpendicular to it allows you to say which way a plane is tilted in space. The normal vector is not unique — multiplying the vector by a positive or negative (but not zero) scalar will not change the orientation of a perpendicular plane.

Now if we think about all the planes perpendicular to a given (or fixed) normal vector, then we can see that we could pick out one of these many planes by knowing one point on the plane.

Say $\mathbf{n}$ is a normal vector to our plane. How do we come up with an equation that is satisfied by all the points on the plane?

The answer to this is a little easier to understand if we think about the case where the origin $(0, 0, 0)$ is one point on the plane. Say $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ is the normal vector and we think of any point $P = (x, y, z)$ on the plane.

In terms of position vectors, you should be able to see fairly easily that the position vector $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ should lie in the plane. So it should be perpendicular to $\mathbf{n}$. But a way to express that by an equation is

$$\mathbf{n} \cdot \mathbf{P} = 0.$$

Writing that out using components we get

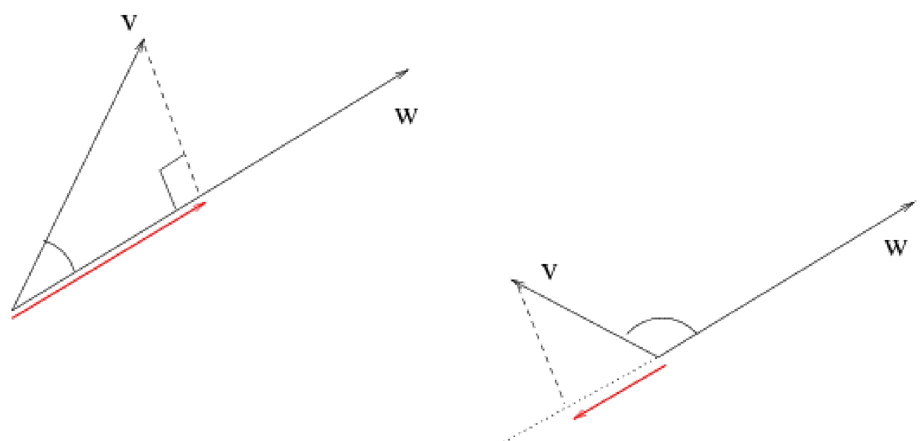
$$n_1 x + n_2 y + n_3 z = 0$$

for the equation of a plane *through the origin* $(0, 0, 0)$. You can see that $x = 0$, $y = 0$, $z = 0$ is one solution of this equation. The coefficients of x, y and z give the components of the normal vector $\mathbf{n}$, and there are no terms with x^2, xy, z^3, etc. It is a linear equation (actually called a *homogeneous linear equation* because of the constant term being 0).

In §1.14, we will return to planes in general, those that don't go through the origin (or need not go thorugh the origin).

1.13 Projections of a vector along another vector

We now define the idea of the orthogonal projection of a vector $\mathbf{v}$ along the direction of a vector $\mathbf{w}$. If you draw a line parallel to $\mathbf{w}$ though the beginning point of $\mathbf{v}$, then the projection of $\mathbf{v}$ along $\mathbf{w}$ is what you might call the shadow that $\mathbf{v}$ makes. Drop a perpendicular from the end of $\mathbf{v}$ to the line just mentioned. That will be the end point of the projection.



The 'length' of the projection will be $\|\mathbf{v}\| \cos\theta$ where θ is the angle between the two vectors. At least that is the length in the case when the angle θ is an acute angle.

To find the projection vector, we need to multiply $\mathbf{w}$ by the right factor to make its length $\|\mathbf{v}\| \cos\theta$. It is easier to start by making a unit vector with the same direction as $\mathbf{w}$. We can do that by dividing $\mathbf{w}$ by its length $\|\mathbf{w}\|$ (or multiplying by $1/\|\mathbf{w}\|$). So

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w}$$

is the unit vector in the direction of $\mathbf{w}$. Then the projection is

$$\|\mathbf{v}\| \cos\theta \left(\frac{1}{\|\mathbf{w}\|}\right)\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}\left(\frac{1}{\|\mathbf{w}\|}\right)\mathbf{w}$$

This formula is also correct in the case of obtuse angles θ, when $\cos\theta < 0$, when the projection will be in the opposite direction to $\mathbf{w}$.

Tidying up the above, we get that the projection along $\mathbf{w}$ of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

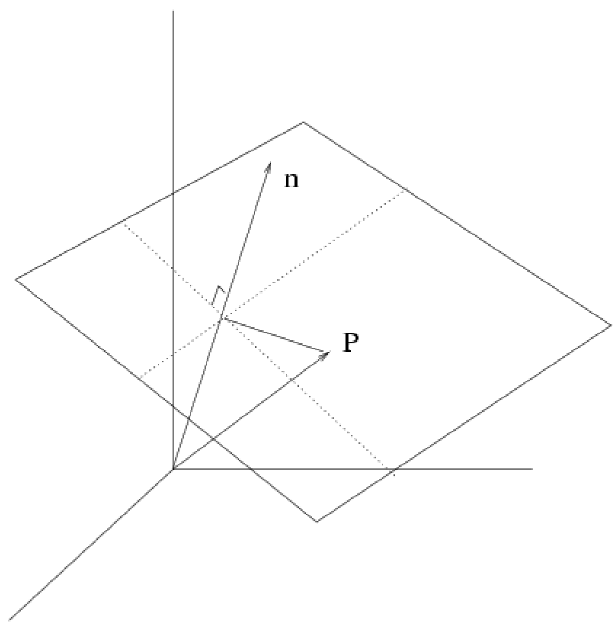
1.14 Equations of planes (in general)

We continue now from where we stopped in §1.12.

To deal with a plane that does not go through the origin, we can see what to do by using projections. Take any plane perpendicular to $\mathbf{n}$. If you look geometrically, you will see that the vector projection

$$\mathrm{proj}_{\mathbf{n}}(\mathbf{P})$$

must be the same no matter what point P in the plane you try. This sketch shows a plane, with its normal vector drawn from the origin though the plane. Then the projection along the direction of the normal $\mathbf{n}$ of the position vector of any point P on the plane is always the same:



That means

$$\frac{\mathbf{P}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}$$

is always the same vector for all points P on the plane. And that translates into a single equation

$$\mathbf{P}\cdot\mathbf{n} = \mathrm{const}$$

If we write that out in components of $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ you get the equation

$$xn_1 + yn_2 + zn_3 = \mathrm{const}$$

which I prefer to write

$$n_1 x + n_2 y + n_3 z = \text{const}$$

In summary then, a plane in space has an equation

$$\boxed{ax + by + cz = d}$$

where $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a (fixed) nonzero vector perpendicular to the plane and d is a constant.

Examples

(i) Find the equation of the plane in space perpendicular to $4\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$ and going through the point $(2, 4, 0)$.

Solution: $\mathbf{n} = 4\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$ is a normal vector to the plane and so the equation has the form

$$4x - 3y + 9z = \text{const}$$

The point $(x, y, z) = (2, 4, 0)$ must satisfy the equation and so we get

$$4(2) - 3(4) + 9(0) = \text{const}$$

or $\text{const} = -4$. So the equation is

$$4x - 3y + 9z = -4$$

(By the way we could multiply the equation across by a nonzero number and get an equally valid equation for the same plane. So $8x - 6y + 18z = -8$ is also correct.)

(ii) Find the equation of the plane in space going through the 3 points $(2, 0, 0)$, $(0, -1, 0)$ and $(0, 0, 7)$.

Solution: We know the equation has the form

$$ax + by + cz = d$$

for some constants a, b, c and d (with a, b, c not all zero). So the unknowns are a, b, c and d.

We have 3 equations for the 4 unknowns because of the 3 points we know have to be on the plane, and they give

$$\left\{ \begin{array}{rrrcl} 2a & & & = & d \\ & -b & & = & d \\ & & 7c & = & d \end{array} \right.$$

We don't seem to have enough information to find all 4 unknowns but what we see is that we can work out a, b and c if we know d, like this:

$$\begin{aligned} a &= d/2 \\ b &= -d \\ d &= d/7 \end{aligned}$$

The question then is what d is?

In fact it does not matter as long as it is not zero. If we choose $d = 1$ we would get $a = 1/2$, $b = -1$ and $c = 1/7$ so that our equation would be

$$\frac{1}{2}x - y + \frac{1}{7}z = 1$$

If we miltiplied that by 14 (which would not change the solutions) we would get

$$7x - 14y + 2z = 14$$

(and you might like that better since it has no fractions).

We could also get this by taking $d = 14$ rather than $d = 1$.

So perhaps you can see that taking a different d multiplies the equation across by something, but the equation you get will still be an equation for a (the) plane through the 3 points given. (Well as long as you don't pick $d = 0$ because then he equation would collapse to $0 = 0$.)

Note also that we were 'lucky' here that the points were on the axes and the 3 simulataneous equations we got were easy to figure out. We'll learn soon an efficient method to solve linear simultaneous equations with possibly large numbers of unknowns. When we know that we'll be able to do more complicated points. As long as 3 points in space are not collinear, there should be just one plane that goes through them.

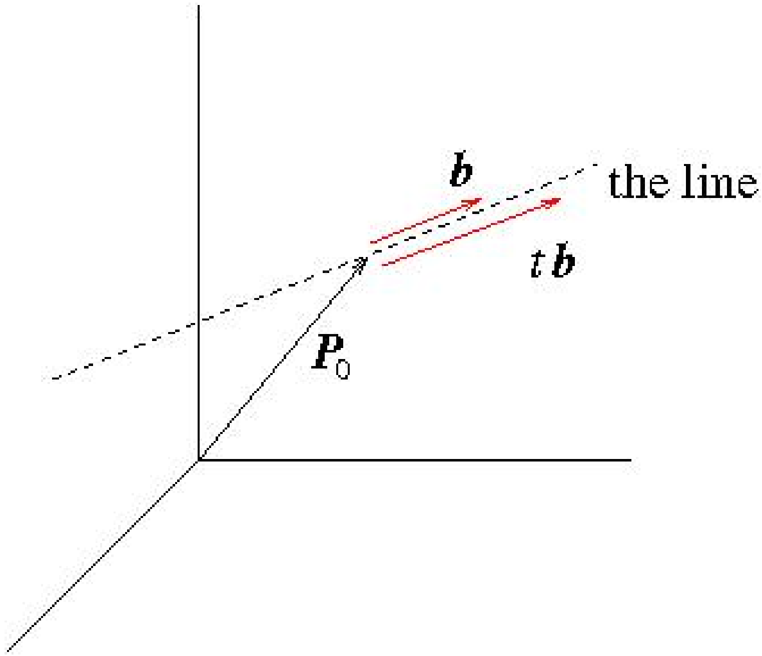
1.15 Lines in space

To describe lines in space, we could try to describe lines as the intersection of two planes (not parallel planes). But, starting with any line, there are many planes that contain the line. If you take one plane containing the line and rotate it around the line, you get all the other planes containing that line.

So if you want to represent the line as the intersection of two planes, there are many possibilities. For that reason it is simpler to go about it a different way at first. We make use of vector methods.

We can describe the orientation of a line in space by giving a (nonzero) vector $\mathbf{b}$ parallel to the line. That will not describe the specific line as any parallel line would be parallel to the same vector $\mathbf{b}$. If we know one point $P_0 = (x_0, y_0, z_0)$ on the line, as well as a vector parallel to the line, we should know the line.

Here is a picture of a line is space, the position vector of one point on the line and a vector $\mathbf{b}$ (not the zero vector) parallel to the line.



Then we can quite easily see that the points with position vectors

$$\mathbf{x} = \mathbf{P}_0 + t\mathbf{b}$$

all lie on the line through the point $\mathbf{P}_0$ in the direction parallel to $\mathbf{b}$. As t varies through all of $\mathbb{R}$ (positive, zero and negative values) we get all the points on that line (once).

We refer to t as a *parameter* and the above equation $\mathbf{x} = \mathbf{P}_0 + t\mathbf{b}$ as a parametric equation for the line in vector form.

If $\mathbf{P}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ = the position vector of the point $P_0 = (x_0, y_0, z_0)$ on the line, if $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and if we write $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then what we have is that the position vectors of the points on the line are

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

This corresponds to 3 scalar equations

$$\left\{ \begin{array}{lcl} x & = & x_0 + b_1 t \\ y & = & y_0 + b_2 t \\ z & = & z_0 + b_3 t \end{array} \right.$$

We call these the parametric equations for the line.

To remember how they work, notice that the right hand side is linear in t, the constant terms come from the coordinates (x_0, y_0, z_0) of a point on the line, and the coefficients of t give the components of a vector $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ parallel to the line.

Examples

(i) Find parametric equations for the line in space that goes through the point $(5, 6, 7)$ and is parallel to the vector $\mathbf{i} - 2\mathbf{j}$.

Solution: We want to take $(x_0, y_0, z_0) = (5, 6, 7)$ and $(b_1, b_2, b_3) = (1, -2, 0)$. We get

$$\begin{cases} x &= 5 + t \\ y &= 6 - 2t \\ z &= 7 \end{cases}$$

for the parametric equations of the line.

(ii) Find parametric equations for the line of intersection of the two planes

$$\begin{array}{rcrcrcr} 2x & & & + & 4z & = & 1 \\ & & y & + & 2z & = & 3 \end{array}$$

Solution: We have three unknowns in these two equatins and so we can't solve them properly. When you think about them as planes that should not surprise you.

One plane is normal to $2\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$ while the other is normal to $0\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. That means they are not parallel planes and so they will intersect in a line. A line has infinitely many points on it.

However, one way to view these equations is that the first tell you x (if you know z), or x in terms of z and the second tell you y in terms of z:

$$\begin{aligned} x &= \frac{1}{2} - 2z \\ y &= 3 - 2z \end{aligned}$$

Chossing different z gives different points (x, y, z). In fact we can think of z as a parameter, something we are free to choose.

If we give z another name t, then we get

$$\begin{aligned} x &= \frac{1}{2} - 2t \\ y &= 3 - 2t \\ z &= t \end{aligned}$$

We have parametric equations now.

(It is not asked, but note that $(1/2, 3, 0)$ is a point on the line and $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is a vector parallel to the line. The point is the one where $t = 0$ and the vector parallel comes from the coefficients of t.)

1.16 Cross products

This is something that makes sense in three dimensions only. There is no really similar product of two vectors in the plane (or there is if we treat them as vectors in space, but the result will be in space). We might have started with dot products in 2 dimensions, then extended the notion to 3 dimensions, but cross products are peculiar to space.

The definition is that the *cross product* $\mathbf{v} \times \mathbf{w}$ of two vectors

$$\begin{aligned} \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \\ \mathbf{w} &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}, \end{aligned}$$

is

$$\boxed{\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_2)\mathbf{k}}$$

There is a sort of 'easy' way to remember the formula. Write a table

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{array}$$

and extend it my repeating the first two columns to the right, like this

$$\begin{array}{ccccc} v_1 & v_2 & v_3 & v_1 & v_2 \\ w_1 & w_2 & w_3 & w_1 & w_2 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \end{array}$$

Then take 'forward diagonal' products with plus signs and backwards diagonals with minus in front

$$\begin{array}{ccccccccccc} & v_1 & & v_2 & & v_3 & & v_1 & & v_2 & \\ & & \searrow & & \swarrow\!\!\!\!\!\!\searrow & & \swarrow\!\!\!\!\!\!\searrow & & \swarrow & & \\ & w_1 & & w_2 & & w_3 & & w_1 & & w_2 & \\ & & \swarrow & & \swarrow\!\!\!\!\!\!\searrow & & \swarrow\!\!\!\!\!\!\searrow & & \searrow & & \\ & \mathbf{i} & & \mathbf{j} & & \mathbf{k} & & \mathbf{i} & & \mathbf{j} & \\ \swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow \\ -v_3w_2\mathbf{i} & & -v_1w_3\mathbf{j} & & -v_2w_1\mathbf{k} & & & v_1w_2\mathbf{k} & & v_2w_3\mathbf{i} & v_3w_1\mathbf{j} \end{array}$$

and add these vectors.

There is actually a pattern[2] to this last formula and so it is not quite impossible to remember.

[2] The first component of $\mathbf{v} \times \mathbf{w}$ depends on the components of $\mathbf{v}$ and $\mathbf{w}$ other than the first. Starting with $v_2w_3 - v_3w_2$ we can get to the next component by adding 1 to the subscripts and interpreting $3 + 1$ as 1. Or think in terms of cycling the subscripts around $1 \to 2 \to 3 \to 1$ to get the next component. You still have to remember the first one.

1.17 Properties of cross products (in $\mathbb{R}^3$)

(i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.

(ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$

Proof. This is not hard. Write both down and you'll see that the same products occur but with opposite signs. □

(iii) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Proof.

$$\begin{aligned} &\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) \\ &\quad = (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \cdot ((v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\ &\quad = v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) \\ &\quad = v_1v_2w_3 - v_1v_3w_2 + v_2v_3w_1 - v_2v_1w_3 + v_3v_1w_2 - v_3v_2w_1 \\ &\quad = 0 \end{aligned}$$

because if you look carefully everything cancels with something else.

So $\mathbf{v} \perp \mathbf{v} \times \mathbf{w}$.

To show $\mathbf{w} \perp \mathbf{v} \times \mathbf{w}$, we can either repeat a similar calculation or we can use $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \perp \mathbf{w}$. □

(iv) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta\|$ where $\theta =$ the angle between $\mathbf{v}$ and $\mathbf{w}$.

Proof. The proof for this is a calculation using

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$\sin^2\theta = 1 - \cos^2\theta$ and multiplying out

$$\|\mathbf{v}\|\|\mathbf{w}\|^2 = (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_2)^2$$

to show it is the same as

$$\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \sin^2\theta = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \cos^2\theta = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2$$

It is not real hard to do the required algebra, but a bit messy. □

(v) Now that we know in a geoemtrical way the length of $\mathbf{v} \times \mathbf{w}$, and we also know that it is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, we can try to describe cross products in a geometrical way.

If the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is not 0 and not π, then the vectors $\mathbf{v}$ and $\mathbf{w}$ are not in the same direction and also not in exactly opposite directions. So as long as $0 < \theta < \pi$, then we can say that there is one plane through the origin parallel to both $\mathbf{v}$ and $\mathbf{w}$ (or containing both vectors if we draw them from the origin). The cross product is then in one of the two normal directions to that plane.

If $\theta = 0$ or $\theta = \pi$, there is no one plane containing $\mathbf{v}$ and $\mathbf{w}$, but in these cases $\sin\theta = 0$ and so we know $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

In the case $0 < \theta < \pi$, we can describe the cross product up to one of the two normal directions to the plane. The question then is to say which direction it is in. If we can identify the top (or upwards) side of the plane somehow, is the cross product pointing up or down? And if the plane is vertical? The anwer to this depends on having the axes fixed in such a way that the direction of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey a 'right-hand rule'. This can be described in terms of the directions of the index finger, first finger and thumb on your right hand if you hold them perpendicular to one another. Another way is to place a cokscrem (an ordinary right-handed corkscrew) along the vertical axis and twist the screw from the x-axis towards the y-axis. It should travel in the direction of the positive z-axis.

For two vectors $\mathbf{v}$ and $\mathbf{w}$, the direction of $\mathbf{v} \times \mathbf{w}$ is described by a right-hand rule. Imaging a corkscrew placed so it is perpendicular to the plane of $\mathbf{v}$ and $\mathbf{w}$. Turn the screw from $\mathbf{v}$ towards $\mathbf{w}$ and the direction it travels is the same as the direction of $\mathbf{v} \times \mathbf{w}$.

(vi) There are some algebraic properties of the cross product that are as you would expect for products:

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \\
(\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \\
(k\mathbf{v}) \times \mathbf{w} &= k(\mathbf{v} \times \mathbf{w}) \\
&= \mathbf{v} \times (k\mathbf{w})
\end{aligned}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and any scalar $k \in \mathbb{R}$. (But recall that the order matters since $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.)

These properties are quite easy to check out.

(vii) Vector triple products

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

make sense, but that is usually not the same as $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

For example

$$\begin{aligned}
\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) &= \mathbf{i} \times \mathbf{k} \\
&= -\mathbf{j} \\
(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} &= \mathbf{0} \times \mathbf{j} \\
&= \mathbf{0}
\end{aligned}$$

1.17.1 Example. Find the equation of the plane that goes through $(1, 2, 3)$, $(3, 1, 2)$ and $(2, 3, 1)$. *Solution:* Let $P = (1, 2, 3)$, $Q = (3, 1, 2)$ and $R = (2, 3, 1)$ and use the same letters for the position vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. Then we can notice that

$$\begin{aligned}
\vec{PQ} &= \mathbf{Q} - \mathbf{P} \\
&= (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\
&= 2\mathbf{i} - \mathbf{j} + \mathbf{k} \\
\vec{PR} &= \mathbf{R} - \mathbf{P} \\
&= (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} - 2\mathbf{k}
\end{aligned}$$

are two vectors that are in the plane we want (or parallel to it). So their cross product must be normal to the plane:

$$\vec{PQ} \times \vec{PR} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

So the plane we want has an equation

$$3x + 3y + 3z = \text{ const.}$$

and we can plug in any of the points P, Q or R to see that the constant has to be 18. Thus the equation of the plane is

$$3x + 3y + 3z = 18$$

or rather this is one possible equation. We can multiply or divide this equation by any nonzero number and still have an equation for the plane. A tidier-looking equations is

$$x + y + z = 6$$

(In retrospect maybe we could have guessed the equation, because the 3 points P, Q and R had the same coordinates permuted around. But the method we used would work for any 3 points, as long as they did not lie in a line.)

1.17.2 Example. Find parametric equations for the line of intersection of the two planes

$$\begin{array}{rcrcrcl}
2x & - & y & + & z & = & 1 \\
x & + & y & - & 2z & = & 3
\end{array}$$

(via cross products).

Solution: We need a vector parallel to the line (and also a point on the line) to write down the parametric equations.

We can think of the vector parallel as being along the line, hence perpendicular to the normal directions for each plane. So we want a vector perpendicular to both $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. But the cross product of those two vectors will work just fine for that and so a vector parallel to the line is

$$(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

So our line will have parametric equations of the form

$$\left\{ \begin{array}{lcl} x &=& x_0 + 3t \\ y &=& y_0 + 3t \\ z &=& z_0 + 3t \end{array} \right.$$

We need to find a point (x_0, y_0, z_0) on the line (any point on it). One way is to look for a point with $z = 0$. We then need x and y coordinates to satisfy

$$\begin{array}{rcrcl} 2x &-& y &=& 1 \\ x &+& y &=& 3 \end{array}$$

Add those to get $3x = 4$ or $x = 4/3$ and then $y = 3 - x = 5/3$.

So our line will have parametric equations

$$\left\{ \begin{array}{lcl} x &=& \frac{4}{3} + 3t \\ y &=& \frac{5}{3} + 3t \\ z &=& 3t \end{array} \right.$$

1.18 Cartesian equations of lines

We started our discussion of lines by saying that using two planes to describe the line is not so convenient, because there are so any possible pairs of planes that intersect in the same line. There is however a way to pick two planes in a standard way. Starting with parametric equations for a line

$$\left\{ \begin{array}{lcl} x &=& x_0 + b_1 t \\ y &=& y_0 + b_2 t \\ z &=& z_0 + b_3 t \end{array} \right.$$

we can solve each of the three equations for the parameter t in terms of all the rest of the quantities. We get

$$t = \frac{x - x_0}{b_1}, \quad t = \frac{y - y_0}{b_2}, \quad t = \frac{z - z_0}{b_3}.$$

(At least we can do this if none of b_1, b_2 or b_3 is zero.) If (x, y, z) is a point on the line, there must be one value of t that gives the point (x, y, z) from the parametric equations, and so the 3 values for t must coincide. We get

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}.$$

These equations are called *cartesian equations for the line*.

You might wonder what sort of equation we have with two equalities in it? Well it is two equations

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} \text{ and } \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}.$$

Each of these two equations is a linear equation, and so the equation of a plane. So we are representing the line as the intersection of two particular planes when we write the cartesian equations.

The first plane

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2}$$

could be rewritten

$$\frac{1}{b_1}x - \frac{1}{b_2}y + 0z = \frac{x_0}{b_1} - \frac{y_0}{b_2}$$

so that it looks like $ax + by + cz = d$. We see that there is no z term in the equation, or the normal vector $(1/b_1, -1/b_2, 0)$ is horizontal. This means that the plane is parallel to the z-axis (or is the vertical plane that contains the line we started with).

We could similarly figure out that the second plane

$$\frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}$$

is the plane parallel to the x-axis that contains our line.

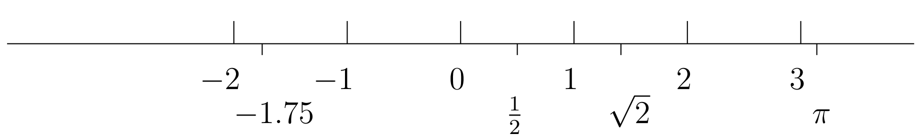
Now, there is an aspect of this that is a bit untidy. From the cartesian equations we can see that

$$\frac{x - x_0}{b_1} = \frac{z - z_0}{b_3}$$

follows immediately, and this is the equation of the plane parallel to the y-axis that contains our line. It seems unsatisfactory that we should pick out the two planes parallel to the z-axis and the x-axis that contains our line, while discriminating against the y-axis in a rather arbitrary way. So although the cartesian equations reduce to representing our line as the intersection of two planes, we could take the two planes to be any two of the planes containing the line and parallel to the x, y and z-axes.

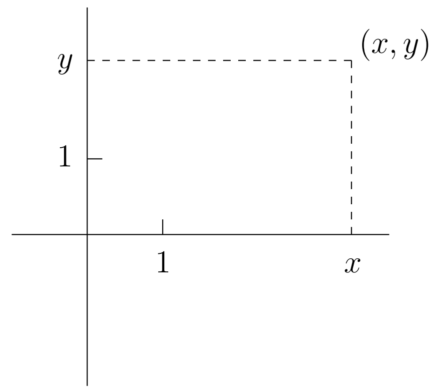
1.19 Higher dimensions

We will use the notation $\mathbb{R}$ (double backed capital R) as a special symbol to signify the set of all real numbers. We are used to picturing $\mathbb{R}$ as the set of all points on an axis or number line.



You should realise that the idea is that the line is all filled up by the real numbers (no gaps). Each real number has its point on the line to represent it and each point corresponds to some number. This is one dimension. We know how to add and multiply numbers.

By $\mathbb{R}^2$ we mean the set of all ordered pairs (x, y) of real numbers and we have a graphical (or pictorial) way of looking at this as the set of all points in a plane.



We can alternatively think of vectors $x\mathbf{i} + i\mathbf{j}$ in two dimensions (as a picture of $\mathbb{R}^2$). Using mathematical set notation

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

(which reads as "R two equals the set of all ordered pairs (x, y) such that x and y are elements of the set $\mathbb{R}$, the set of real numbers").

By $\mathbb{R}^3$ we mean the set of ordered triples (x, y, z) or real numbers. That is

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

and we are now used to the idea that these triples can be pictured by either looking at points (x, y, z) in space, or looking at vectors (arrows) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space. We can think in terms of coordinates of points, or of the components of vectors, and we have on several occasions realised that it can be convenient to switch from a point to its position vector. When we manipulate vectors, we can think graphically about what we are doing, but it is usually much easier to calculate via components.

When you look at the progression from $\mathbb{R}$ to $\mathbb{R}^2$ to $\mathbb{R}^3$ formally, from real numbers to ordered pairs of real numbers to ordered triples, there does not seem to be anything to stop us going on to ordered 4-tuples (x_1, x_2, x_3, x_4) of real numbers, or to ordered 5-tuples $(x_1, x_2, x_3, x_4, x_5)$. Mathematically we can just define

$$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

and

$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\}$$

In fact, for any $n = 2, 3, 4, \ldots$ we let $\mathbb{R}^n$ be the set of all n-tuples of real numbers. By an n-tuple $(x_1, x_2, \ldots, x_n)$ we mean a list of n numbers where the order matters. So

$$(1, 2, 3, 4, 5) \in \mathbb{R}^5$$

is different from

$$(2, 1, 3, 4, 5) \in \mathbb{R}^5.$$

In general then, two n-tuples

$$(x_1, x_2, \ldots, x_n)$$

and

$$(y_1, y_2, \ldots, y_n)$$

are to be considered equal only when they are absolutely identical. That means only when

$$x_1 = y_1, \text{ and } x_2 = y_2, \text{ etc, up to } \ldots, \text{ and } x_n = y_n.$$

Points in different dimensions should not be compared. So $(1, 2) \in \mathbb{R}^2$ and $(1, 2, 0) \in \mathbb{R}^3$ are not the same.

Question: Why do we need $\mathbb{R}^4$, $\mathbb{R}^5$, and so on?

There is no good way to picture them. We can't really draw or even think in 4 dimensions, not in any satisfactory way. And 5 dimensions is even worse from that point of view. Nevertheless, there are many practical reasons why $\mathbb{R}^n$ is useful even when n is very large.

One example, relating directly to what we did at the start of the course, concerns systems of linear equations. There are many applications for these systems of linear equations, often involving large numbers of unknowns. If there are (say) 5 unknowns x_1, x_2, x_3, x_4, x_5 it turns out to be convenient to think of the solutions as points $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$.

What sort of applications involve 5 unknowns (or more)? Here is a simple example. Suppose you are working for the consumer protection agency and you want to make sure that the milk sold in the shops is in accordance to what it claims to be on the label. What you might do is gather up a number of cartons of milk (say we stick to one size and brand) and do various measurements on each of those you collected. Maybe

- volume of milk in the carton
- fat content
- calcium content
- temperature of the milk in the shop
- vitamin D

So, you would end up with 5 numbers for each carton of milk. It is not a big step to consider these numbers as giving one point in $\mathbb{R}^5$ for each sample. The kinds of techniques you would use to analyse the data are most naturally described in terms of manipulating points in $\mathbb{R}^5$.

We are now going to describe some of the basic kinds of manipulation we can do on points in $\mathbb{R}^5$. They will be exact parallels to what we did for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

It may seem strange to treat points like vectors, but remember we already got into the habit of changing from points to their position vectors. And we do most calculations with the numbers, the components of the vectors. Finally, we can't see what we are doing in higher dimensions and so there is no reason to distinguish between points and vectors.

1.20 Arithmetic in $\mathbb{R}^n$

In $\mathbb{R}^n$ we define the *sum* of two n-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$$

to be

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

If $k \in \mathbb{R}$ is a scalar and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ then we define $k\mathbf{x}$ by the rule

$$k\mathbf{x} = (kx_1, kx_2, \ldots, kx_n).$$

If you think about it, you will realise that we already did exactly these operations when we were doing row operations on matrices. Rows are just n-tuples for some n and we described row operations by a less formal method than what we have just done now. But when we added rows, we were using these same operations.

Example: If $\mathbf{x} = (1, 2, 3, 4)$ and $\mathbf{y} = (7, 8, 9, 10)$ are in $\mathbb{R}^4$, then

$$\begin{aligned}
5\mathbf{x} &= (5, 10, 15, 20) \\
2\mathbf{y} &= (14, 16, 18, 20) \\
5\mathbf{x} + 2\mathbf{y} &= (5 + 14, 10 + 16, 15 + 18, 20 + 20) \\
&= (19, 26, 33, 40)
\end{aligned}$$

We can see that there is no real problem following the rules to calculate these operations.

It is possible to check out that the operations on $\mathbb{R}^n$ that we have defined obey the 'standard' rules of algebra. We won't spell that out, or check out that those rules are really valid, but they are. So things like $9(\mathbf{x} + \mathbf{y}) = 9\mathbf{x} + 9\mathbf{y}$ are true for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

1.21 Dot products in $\mathbb{R}^n$

We *define* the dot product of two elements (vectors or points as you like)

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$$

to be the number

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

So this is the same rule as we know from $\mathbb{R}^2$ and $\mathbb{R}^3$ extended to take in all the coordinates.

We define the *magnitude* of $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as

$$\begin{aligned}
\|\mathbf{x}\| &= \sqrt{\mathbf{x} \cdot \mathbf{x}} \\
&= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\end{aligned}$$

We define the *distance* between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$\begin{aligned}
\mathrm{distance}(\mathbf{x}, \mathbf{y}) &= \|\mathbf{y} - \mathbf{x}\| \\
&= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}
\end{aligned}$$

1.21.1 Theorem (Cauchy-Schwarz inequality)**.** *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,*

$$-\|\mathbf{x}\|\|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

always holds.

We can state this as

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

We are not going to prove that this is true.

Note however though that in $\mathbb{R}^2$ and $\mathbb{R}^3$ we showed that

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. We can't use that in $\mathbb{R}^n$ when $n \geq 4$ because we can't see anything in higher dimensions. What we do is go partly by analogy with the case of the plane and 3-space. But we do need proofs of theorems like the above that don't rely on imagining triangles in higher dimensional space. (At least, ideally we should justify the theorem by giving a proof of it. But we won't.)

Now that we have the inequality, we can make a definition of an angle.

1.21.2 Definition. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define the angle θ between $\mathbf{x}$ and $\mathbf{y}$ to be that $\theta \in [0, \pi]$ where

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$$

There is a snag in this definition. It really says

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

and there is a potential problem of division by 0. That problem happens only when one of the points $\mathbf{x}$ or $\mathbf{y}$ has all coordinates 0. The point

$$\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$$

(where we mean there are n zeros) is called the *zero vector* or the *origin* in $\mathbb{R}^n$.

Example. Find the cosine of the angle between $\mathbf{x} = (1, -1, 2, 1)$ and $\mathbf{y} = (2, 1, 0, 2)$ (in $\mathbb{R}^4$).

Solution.

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
&= \frac{(1)(2) + (-1)(1) + (2)(0) + (1)(2)}{\sqrt{1^2 + (-1)^2 + 2^2 + 1^2}\sqrt{2^2 + 1^2 + 0^2 + 2^2}} \\
&= \frac{3}{\sqrt{7}\sqrt{9}} = \frac{1}{\sqrt{7}}
\end{aligned}$$

1.21.3 Definition. We call $\mathbf{x} \in \mathbb{R}^n$ *perpendicular* (or *orthogonal*) to $\mathbf{y} \in \mathbb{R}^n$ if $\mathbf{x} \cdot \mathbf{y} = 0$.

Note that this definition is the same as having angle $\theta = 0$ **except** that that we now define the origin to be perpendicular to every other vector. Or, in other words, the zero vector is perpendicular to every other vector.

Proceeding still further by analogy with the case of planes in space, we now make a terminology for 'hyperplanes' in $\mathbb{R}^n$.

1.21.4 Definition. If $\mathbf{N} = (N_1, N_2, \ldots, N_n)$ is a nonzero vector in $\mathbb{R}^n$, and if $c \in \mathbb{R}$ is a constant, then the set of solutions $\mathbf{x} \in \mathbb{R}^n$ to the equation

$$\mathbf{N} \cdot \mathbf{x} = c$$

is called a *hyperplane* in $\mathbb{R}^n$.

If we write out the equation using coordinates for the point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ on the plane, the equation becomes

$$N_1 x_1 + N_2 x_2 + \cdots + N_n x_n = c$$

We call $\mathbf{N}$ a *normal vector* to the hyperplane.

We won't go any further with $\mathbb{R}^n$ for now, but it will arise again.