**2E1 (Timoney) Tutorial sheet 9**
[Tutorials December 6 – 7, 2006]

**Name:** Solutions

1. Find the local maximum and local minimum points of the function

$$h(x, y) = 5xy - 7x^2 + 3x - 6y.$$

*Solution:* To find the critical points, set both partials of $h$ equal 0.

$$\begin{aligned} \frac{\partial h}{\partial x} &= 5y - 14x + 3 \\ \frac{\partial h}{\partial y} &= 5x - 6 \\ 5x - 6 &= 0 \qquad \text{gives} \\ x &= \frac{6}{5} \\ 5y - 14x + 3 &= 0 \\ 5y - 14\frac{6}{5} + 3 &= 0 \\ 5y &= \frac{84 - 15}{5} = \frac{69}{5} \\ y &= \frac{69}{25} \end{aligned}$$

Thus there is only one critical point $(\frac{6}{5}, \frac{69}{25})$. For the second derivative test, we calculate

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} &= -14 & \frac{\partial^2 h}{\partial x \partial y} &= 5 \\ \frac{\partial^2 h}{\partial x \partial y} &= 5 & \frac{\partial^2 h}{\partial y^2} &= 0 \end{aligned}$$

Thus

$$\Delta = \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 h}{\partial y^2} - \left(\frac{\partial^2 h}{\partial x \partial y}\right)^2 = (-14)(0) - 5^2 = -25 < 0.$$

Thus the critical point is a saddle point. There are no local max or min points for $h$.

2. Find the points on the surface $z^2 - xy = 4$ closest to the origin. [Hint: Calculations are easier if you minimize $f(x, y, z) = x^2 + y^2 + z^2 =$ the square of the distance to the origin, rather than its square root.]

*Solution:* We can think of this as a Lagrange multiplier problem. Let $f(x, y, z) = x^2 + y^2 + z^2$. (The minimum distance and the minimum value of $f$ will be at the same point.) If we take $g(x, y, z) = z^2 - xy - 4$, we have the constraint $g(x, y, z) = 0$ and so Lagrange multipliers says that at the minimum (if there is one) we should have $\nabla f = \lambda \nabla g$ as well as $g(x, y, z) = 0$. This gives us a system of 4 equations

$$\begin{cases} (2x, 2y, 2z) &= \lambda(-y, -x, 2z) \\ z^2 - xy - 4 &= 0 \end{cases}$$

From the equation $2z = 2\lambda z$ we see that $(1 - \lambda)z = 0$ and so either $\lambda = 1$ or $z = 0$.

If $\lambda = 1$, we have to have $2x = -y$ and $2y = -x$ so that $x = -y/2 = x/4$. That forces $x = y = 0$ and so then we get $z = \pm 2$ from $z^2 - xy - 4 = 0$. That leaves us with the two points $(0, 0, \pm 2)$ to consider. Both are distance 2 from the origin.

If $z = 0$, then we have $2x = -\lambda y$ and $2y = -\lambda x$. These give $x = (-\lambda/2)y = (-\lambda/2)^2 x$. So either $\lambda^2/4 = 1$ or we must have $x = 0$. We can rule out $x = 0$ because we would then have $y = (-\lambda/2)x = 0$ and $(0, 0, 0)$ does not satisfy $z^2 - xy - 4 = 0$. So we are left with $\lambda = \pm 2$. If $\lambda = 2$ then $x = -y$ and from $z^2 - xy - 4 = 0$ we get $0 + x^2 = 4$, so $x = \pm 2$. We get the points $(\pm 2, \mp 2, 0)$ (which are at distance $2\sqrt{2}$ from the origin).

Finally, if $z = 0$ and $\lambda = -2$ we have $x = y$ and from $z^2 - xy - 4 = 0$ we get $0 - x^2 - 4 = 0$, which can't be.

The only possible points then are $(0, 0, \pm 2)$ and $(\pm 2, \mp 2, 0)$. Since it stands to reason (more or less) that there should be a closest point to the origin, we can say that the closest points are $(0, 0, \pm 2)$.

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