2E1 (Timoney) Tutorial sheet 7
[Tutorials November 22 – 23, 2006]

Name: Solutions

1. Find the equation of the tangent plane to the graph $z = x \sin y$ at the point where $(x, y) = (1, \pi/3)$.

   Solution: We can do this with the linear approximation formula that $z = f(x, y) = x \sin y$ is approximately the same as the linear approximation

$$z = f(1, \pi/3) + \frac{\partial f}{\partial x} |_{(1,\pi/3)} (x-1) + \frac{\partial f}{\partial y} |_{(1,\pi/3)} (y - \pi/3)$$

   (when (x, y) is near $(1, \pi/3)$) and this is the equation of the tangent plane to the graph. We can work out $f(1, \pi/3) = \sin(\pi/3) = \sqrt{3}/2$ and we can calculate

$$\frac{\partial f}{\partial x} = \sin y \qquad \frac{\partial f}{\partial y} = x \cos y$$
$$\frac{\partial f}{\partial x} |_{(1,\pi/3)} = \sin(\pi/3) = \sqrt{3}2 \qquad \frac{\partial f}{\partial y} |_{(1,\pi/3)} = \cos(\pi/3) = 1/2$$

   and so we get that the equation is

$$z = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}(x-1) + \frac{1}{2}\left(y - \frac{\pi}{3}\right)$$

   Alternatively we can say that the graph is the level surface $g(x, y, z) = 0$ to the function $g(x, y, z) = z - x \sin y$ at the point $(1, \pi/3, \sin(\pi/3)) = (1, \pi/3, \sqrt{3}/2)$ and so the tangent plane has the gradient vector $\nabla g |_{(1,\pi/3,\sqrt{3}/2)}$ as normal vector. And it is a plane through the point $(1, \pi/3, \sqrt{3}/2)$. Using $(\alpha, \beta, \gamma) = \nabla g |_{(1,\pi/3,\sqrt{3}/2)}$ we can write that the equation should be

$$\alpha x + \beta y + \gamma z = c$$

   and use the point to find c. Or we can write that the equation is

$$\alpha(x-1) + \beta(y - \pi/3) + \gamma(z - \sqrt{3}/2) = 0.$$

   We need to know the values of the partials of g at the point $(1, \pi/3, \sqrt{3}/2)$ and so we compute

$$\frac{\partial g}{\partial x} = -\sin y \qquad \frac{\partial g}{\partial y} = -x \cos y$$
$$\frac{\partial g}{\partial z} = 1$$
$$\frac{\partial g}{\partial x} |_{(1,\pi/3,\sqrt{3}/2)} = -\sin(\pi/3) = -\sqrt{3}/2 \qquad \frac{\partial g}{\partial y} |_{(1,\pi/3,\sqrt{3}/2)} = -\cos(\pi/3) = -1/2$$
$$\frac{\partial g}{\partial z} |_{(1,\pi/3,\sqrt{3}/2)} = 1$$

So the equation is

$$-\frac{\sqrt{3}}{2}(x-1) - \frac{1}{2}\left(y - \frac{\pi}{3}\right) + 1\left(z - \frac{\sqrt{3}}{2}\right) = 0.$$

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2. Find the equation of the tangent plane to the surface

$$\frac{x^2e^xy + xe^yz}{x+y+z} = \frac{2e}{3}$$

at the point $(1,1,1)$.

Solution: We are getting the tangent plane to a level surface $f(x,y,z) = 2e/3$ where the function $f(x,y,z)$ is given by the left hand side of the equation. So the normal vector to the tangent plane is the gradient of f evaluated at the point $(1,1,1)$. We compute

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(2xe^xy + x^2e^xy + e^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial y} &= \frac{(x^2e^x + xe^yz)(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial z} &= \frac{xe^y(x+y+z) - (x^2e^xy + xe^yz)}{(x+y+z)^2} \\
\frac{\partial f}{\partial x}\,|_{(1,1,1)} &= \frac{(4e)(3) - 2e}{3^2} = \frac{10e}{9} \\
\frac{\partial f}{\partial y}\,|_{(1,1,1)} &= \frac{(2e)(3) - 2e}{3^2} = \frac{4e}{9} \\
\frac{\partial f}{\partial z}\,|_{(1,1,1)} &= \frac{3e - 2e}{3^2} = \frac{e}{9}
\end{aligned}$$

For the tangent plane we get

$$\frac{10e}{9}(x-1) + \frac{4e}{9}(y-1) + \frac{e}{9}(z-1)$$

and we could simplify this by dividing by $e/9$ to get the equivalent equation

$$10(x-1) + 4(y-1) + (z-1) = 0.$$

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