2E1 (Timoney) Tutorial sheet 2
[Tutorials October 18 – 19, 2006]

Name: Solutions

1. Find parametric equations for the line in $\mathbb{R}^3$ that passes through both points $(1, 2, 3)$ and $(3, -2, 1)$.

   Solution: A vector parallel to the line is the difference $(3, -2, 1) - (1, 2, 3) = (2, -4, -2)$. So, using the point $(1, 2, 3)$ and the vector parallel we get paramemtric equations in vector form
$$\mathbf{x} = (1, 2, 3) + t(2, -4, -2)$$
   and we expand these into
$$\begin{cases} x_1 &= 1 + 2t \\ x_2 &= 2 - 4t \\ x_3 &= 3 - 2t \end{cases}$$

2. If a particle moving in space $\mathbb{R}^3$ has position $\mathbf{x}(t) = (t, \cos t, t^3)$ at time t, find its velocity vector, speed, acceleration, and rate of change of speed, all evaluated at time $t = 1$.

   Solution: The velocity is $\mathbf{v}(t) = d\mathbf{x}/dt$, the speed is the magnitude of the vector $\mathbf{v}(t)$, the acceleration is $\mathbf{a}(t) = d\mathbf{v}/dt$ and the rate of change of speed is $\frac{d}{dt}\|\mathbf{v}(t)\|$. We've to calculate all these and set $t = 1$.
$$\begin{aligned}
\mathbf{v}(t) &= \frac{d\mathbf{x}}{dt} \\
&= (1, -\sin t, 3t^2) \\
\|\mathbf{v}(t)\| &= \sqrt{1 + \sin^2 t + 9t^4} \\
\mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} \\
&= (0, -\cos t, 6t) \\
\frac{d}{dt}\|\mathbf{v}(t)\| &= \frac{2\sin t\cos t + 36t^3}{2\sqrt{1 + \sin^2 t + 9t^4}} \\
&= \frac{\sin t\cos t + 18t^3}{\sqrt{1 + \sin^2 t + 9t^4}} \\
\mathbf{v}(1) &= (1, -\sin 1, 6) \\
\|\mathbf{v}(1)\| &= \sqrt{10 + \sin^2 1} \\
\mathbf{a}(1) &= (0, -\cos 1, 6) \\
\frac{d\|\mathbf{v}(t)\|}{dt}\Big|_{t=1} &= \frac{\sin 1\cos 1 + 18}{\sqrt{10 + \sin^2 1}}
\end{aligned}$$

3. For the vector function $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3$ given by $\mathbf{x}(t) = (e^t \cos t, e^t \sin t, 2)$, calculate the unit tangent vector, the unit normal vector and the curvature.

   Solution: We know $\mathbf{T} = \mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$, $\mathbf{N} = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$ is the unit normal and the curvature is $\kappa = \|d\mathbf{T}/ds\| = \|d\mathbf{T}/dt\|/(ds/dt) = \|\mathbf{T}'(t)\|/\|\mathbf{x}'(t)\|$.

$$
\begin{aligned}
\mathbf{x}'(t) &= (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 0)\\
&= e^t(\cos t - \sin t, \sin t + \cos t, 0)\\
\|\mathbf{x}'(t)\| &= e^t \|(\cos t - \sin t, \sin t + \cos t, 0)\|\\
&= e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\\
&= e^t \sqrt{\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\sin t \cos t + \cos^2 t}\\
&= e^t \sqrt{2\cos^2 t + 2\sin^2 t} = e^t\sqrt{2}\\
\mathbf{T}(t) &= (\cos t - \sin t, \sin t + \cos t, 0)/\sqrt{2}\\
&\quad (= \text{ unit tangent vector})\\
\mathbf{T}'(t) &= (-\sin t - \cos t, \cos t - \sin t, 0)/\sqrt{2}\\
\|\mathbf{T}'(t)\| &= \frac{1}{\sqrt{2}}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2}\\
&= \frac{1}{\sqrt{2}}\sqrt{2} = 1\\
\mathbf{N}(t) &= \mathbf{T}'(t)/\|\mathbf{T}'(t)\|\\
&= (-\sin t - \cos t, \cos t - \sin t, 0)/\sqrt{2}\\
&\quad (= \text{ unit normal vector})\\
\kappa &= \|\mathbf{T}'(t)\|/\|\mathbf{x}'(t)\|\\
&= 1/(e^t\sqrt{2}) = e^{-t}/\sqrt{2}\\
&\quad (= \text{ curvature})
\end{aligned}
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