MA2224 (Lebesgue integral) Tutorial sheet 10
[April 4, 2012]

Name: Solutions

1. Let
$$f_n(x) = \chi_{[1/n,1]}(x)\frac{1}{\sqrt{x}}$$

(a) What is $\int_{\mathbb{R}} f_n\,d\mu$ (the Lebesgue integral). [Hint: Theorem 3.7.4]
Solution: It is the same as $\int_{[1/n,1]} f_n\,d\mu$ and (since the integrand coincides with $1/\sqrt{x}$ on $[1/n,1]$) that coincides with the ordinary Riemann integral
$$\int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx$$
We can works that out by getting the antiderivative of the integrand
$$\int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}]_{1/n}^{1} = 2 - \frac{1}{\sqrt{n}}$$

(b) Show that $(f_n)_{n=1}^{\infty}$ is a monotone increasing sequence of functions.
Solution: For $x \in [1/n,1]$ we have $x \in [1/(n+1),1]$ and $f_n(x) = f_{n+1}(x) = 1/\sqrt{x}$. For $x \in [1/(n+1),1] \setminus [1/n,1] = [1/(n+1),1/n)$ we have
$$0 = f_n(x) < \frac{1}{\sqrt{x}} = f_{n+1}(x)$$
Finally for $x \notin [1/(n+1),1]$ we have $f_n(x) = f_{n+1}(x) = 0$. So we have
$$f_n(x) \le f_{n+1}(x) \quad \text{(for all } x \in \mathbb{R})$$
and for all $n \in \mathbb{N}$.
That implies $f_n(x) \le f_m(x)$ for $n \le m$ (all $x \in \mathbb{R}$) and the sequence is monotone increasing.

(c) What is $\int_{\mathbb{R}} f\,d\mu$ where $f(x) = 1/\sqrt{x}$ when $0 < x \le 1$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus (0,1]$? [Hint: Monotone convergence theorem (3.5.6)]
Solution: We have (for each $x \in \mathbb{R}$)
$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$
(because if x satisfies $0 < x \le 1$ then for $n \ge 1/x$ we have $f_n(x) = 1/\sqrt{x}$, thus the limit is $1/\sqrt{x}$, while if $x \in \mathbb{R} \setminus (0,1]$ we have $f_n(x) = 0$ for all n).

The Monotone convergence theorem says

$$\int_{\mathbb{R}} f\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu = \lim_{n\to\infty} 2 - \frac{2}{\sqrt{n}} = 1$$

(because the sequence (f_n) is a monotone increasing sequence of nonnegative measurable functions with [pointwise] limit equal to f).

2. Let $f_n = (1/n)\chi_{[n,2n]}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$,

   $$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu$$

   and

   $$\int_{\mathbb{R}} f\,d\mu.$$

   Solution:

   $$f(x) = \lim_{n\to\infty} f_n(x) = 0$$

   (**either** because $0 \leq f_n(x) \leq 1/n$ for all x **or** because for $n > x$ we have $f_n(x) = 0$).

   $$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu = \lim_{n\to\infty} \frac{1}{n}\mu([n,2n]) = \lim_{n\to\infty} \frac{1}{n}(2n-n) = \lim_{n\to\infty} 1 = 1$$

   $$\int_{\mathbb{R}} f\,d\mu = 0$$

   (b) Why does this not contradict the dominated convergence theorem (4.3.1)?

   Solution: The dominated convergence theorem says that the limit of the integrals $\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu$ coincides with the integral of the limit (which is false here) but the theorem requires the existince of a dominating integrable function. That is there is a hypothesis that there exists $g\colon \mathbb{R} \to \mathbb{R}$ integrable with

   $$|f_n(x)| \leq g(x) \quad \text{(all } x \in \mathbb{R}\text{)}$$

   (and integrable means that g is measurable with $\int_{\mathbb{R}} |g|\,d\mu < \infty$ — in our case $g = |g|$ must hold and so what we need is g with $\int_{\mathbb{R}} g\,d\mu < \infty$).

   So it must be that there is no such g.

   If you like we can show that there is no such g because $g(x) \geq f_n(x)$ implies $g(x) \geq 1/n$ on $[n, n+1)$ and so we can say that g would satisfy

   $$g \geq \sum_{j=1}^{n} \frac{1}{j}\chi_{[j,j+1)}$$

(for each n). Then

$$\int_{\mathbb{R}} g\,d\mu \geq \int_{\mathbb{R}} \sum_{j=1}^{n} \frac{1}{j}\chi_{[j,j+1)}\,d\mu = \sum_{j=1}^{n} \frac{1}{j}\mu([j,j+1)) = \sum_{j=1}^{n} \frac{1}{j}$$

But

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and so $\int_{\mathbb{R}} g\,d\mu$ could not be finite.

That is an exlicit argument why there is no dominating g.

(c) What does Fatou's lemma say about this example?

Solution: Since the functions f_n are nonnegative and measurable Fatou's theorem doies apply and it says

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu \geq \int_{\mathbb{R}} \liminf_{n\to\infty} f_n\,d\mu$$

That is indeed true because

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu = \liminf_{n\to\infty} 1 = 1$$

while

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n\,d\mu = \int_{\mathbb{R}} 0\,d\mu = 0$$

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