UNIVERSITY OF DUBLIN

MA1S13

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Natural Sciences **Trinity Term 2000**
JF Human Genetics
JF Computational Physics & Chemistry
JF Medicinal Chemistry

COURSE: 1S PAPER 3

Tuesday, May 23 Goldsmith Hall 9.30 — 12.30

Dr. R. M. Timoney

Answer 6 questions. Logarithmic tables are available and they contain a table of values of the normal distribution.

Just the answers

1. **(a)** Write out the four (base 10) numbers $3, 7, 19, 31$ in binary and explain how a modern computer might use 32 bits to store these numbers.

   Answer. $3 = (11)_2$, $7 = (111)_2$, $19 = (10011)_2$, $31 = (11111)_2$.

   Explanation ...store binary digits as bits ...on=1/off=0 switches ...normally use 32 bits for each integer with one bit for a sign

3	0	0	...	0	0	0	0	1	1
7	0	0	...	0	0	0	1	1	1
19	0	0	...	0	1	0	0	1	1
31	0	0	...	0	1	1	1	1	1
Bit position:	1	2	...	27	28	29	30	31	32

   (b) Convert the hexadecimal number $(c14b)_{16}$ to octal by first converting it to binary via the "4 binary for one hexadecimal" rule and then using a similar rule to convert to octal.

   Answer. $(c14b)_{16} = (1100\,0001\,0100\,1011)_2 = (001\,100\,000\,101\,001\,011)_2 = (140513)_8$

   (c) Convert $29/5$ to binary. Then convert the binary result to scientific notation and give the (binary) mantissa and exponent for it.

   Answer. $29/5 = 5 + 4/5 = (101.\overline{1100})_2 = (1.01\overline{1100})_2 * 2^2$

   Mantissa $= (1.01\overline{1100})_2$, exponent $= 2 = (10)_2$

2. **(a)** Write a `Mathematica` instruction to define a new function $f(x)$ given by

$$f(x) = \cos(3e^x + 5x)$$

Answer. `f[x_] = Cos[ 3 Exp[x] + 5 x]`

(b) What would you type to ask `Mathematica` to plot the function $f(x)$ just defined in part (a) over the range $0 \le x \le 4$?

Answer. `Plot[f[x], {x, 0, 4}]`

(c) What would you type to ask `Mathematica` to compute the derivative of the function $f(x)$ in part (a)?

Answer. `f'[x]`

(d) What would you type to ask `Mathematica` to plot the parametric curve

$$\begin{aligned} x &= 3\cos t \\ y &= 5\sin t \qquad (0 \le t \le 2\pi). \end{aligned}$$

Answer.

```
ParametricPlot[ { 3 Cos[t], 5 Sin[t]}, {t, 0, 2 Pi} ]
```

3. **(a)** Consider the graph

$$y = \frac{x-1}{x^2 - 5x + 6}$$

Find out where it is increasing, where it is decreasing, its critical points, its local maxima and minima and its asymptotes.

Answer. Critical points at $f'(x) = 0$, that is $x = 1 \pm \sqrt{2}$

Vertical asymptotes at $x^2 - 5x + 6 = 0$, that is $x = 3$ and $x = 2$. Horizontal asymptote at $y = \lim_{x\to\pm\infty} \frac{x-1}{x^2-5x+6}$, that is $y = 0$.

Decreasing $(-\infty, 1-\sqrt{2}]$, $[1+\sqrt{2}, 3)$ and $(3, \infty)$.

Increasing $[1-\sqrt{2}, 2)$, $(2, 1+\sqrt{2}]$.

Local minimum at $1-\sqrt{2}$, local maximum at $1+\sqrt{2}$.

(b) Find the intervals where the graph

$$y = x^4 - 2x^3 - 72x^2 + 36x - 18$$

is concave upwards, the intervals where it is concave downwards and its points of inflection.

Answer. $\frac{d^2y}{dx^2} = 12x^2 - 12x - 144 = 12(x^2 - x - 12) = 12(x-4)(x+3)$. Concave up on $(-\infty, -3]$ and $[4, \infty)$. Concave down on $[-3, 4]$.

4. **(a)** Write down an integral that gives the length of the parametric curve

$$\begin{aligned} x &= 3\cos t \\ y &= 5\sin t \qquad 0 \le t \le 2\pi. \end{aligned}$$

Answer. $\int_0^{2\pi} \sqrt{(-3\sin t)^2 + (5\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{9\sin^2 t + 25\cos^2 t}\, dt$

(b) Write a `Mathematica` command that should find the numerical value of the integral in part (a).

Answer.

```
NIntegrate[ Sqrt[ 9 (Sin[t])^2 + 25 (Cos[t])^2 ], {t,0,2 Pi}]
```

(c) Find the total area of the following region of the plane:

$$\{(x, y) : 0 \le x \le \pi, \cos 2x \le y \le e^x\}.$$

Answer. $e^\pi - 1$

5. **(a)** State the Fundamental Theorem of Integral Calculus (both parts).

   Answer. See the statements in Anton's Calculus, 6th edition, pages 417 and 423.

   (b) Find

$$\frac{d}{dx} \int_{-1}^{x^5} \sin(t^3 + 1)\, dt$$

   Answer. $5x^4 \sin(x^{15} + 1)$

   (c) Suppose $h(x)$ is a function of $x \in \mathbb{R}$ and that it satisfies both

$$h'(x) = \sin(x^3 + 1)$$

   and $h(-1) = 0$. Find a formula for $h(x)$ which involves an integral.

   Answer. $h(x) = \int_{-1}^{x} \sin(t^3 + 1)\, dt$

6. Find the following integrals analytically (that is, without using numerical approximations or computer algebra):

   (a) $\displaystyle\int \sin^3 x \cos^6 x\, dx$

   Answer. $\frac{\cos^9 x}{9} - \frac{\cos^7 x}{7} + C$

   (b) $\displaystyle\int (x^2 - x)e^x\, dx$

   Answer. $x^2e^x - 3xe^x + 3e^x + C$

   (c) $\displaystyle\int_0^{\pi/3} \sec^6 x\, dx.$

   Answer. $24\sqrt{3}/5$

7. Find the following integrals analytically:

   (a) $\displaystyle\int \frac{x + 1}{(x + 2)(x^2 + 2x + 3)}\, dx$

   Answer. $\displaystyle\frac{\sqrt{2}}{3} \tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) - \frac{1}{3} \ln|x + 2| + \frac{1}{6} \ln(x^2 + 2x + 3) + C$

(b) $\displaystyle\int_{-1}^{2} \frac{1}{x^{(4/3)}}\,dx$ (an improper integral).
Answer. Does not converge.

8. **(a)** A bead is made by drilling out a hole of radius 2 (units of length) through the centre of a solid sphere of radius 6. What is the volume of the bead?
Answer. $512\pi\sqrt{2}/3$

(b) A central heating oil tank has the shape of a right circular cylinder with its axis horizontal and radius 0.5 (metres). Find the total pressure force exerted by the oil on one of the vertical circular ends of the tank when the tank is just full of oil (in terms of $\rho =$ the weight density of the oil).
Answer. $\frac{\pi}{8}\rho$

9. **(a)** State the *linear approximation formula* for a function $y = f(x)$ near a point $x = a$ and use it to compare the true value of $f(x) = \sqrt{16 + x^2}$ for $x = 3.3$ with the value you get by using linear approximation centered at $x = 3$.
Answer. The linear approximation formula states that if $f(x)$ is differentiable at $x = a$, then

$$f(x) \cong f(a) + f'(a)(x - a)$$

for x near a (and that the error in the approximation tends to zero as a proportion of $x - a$ as $x \to a$).
By linear approximation to $f(x) = \sqrt{16 + x^2}$ centered on $x = 3$ we have $\sqrt{16 + x^2} \cong 5 + (3/5)(x - 3)$ for x close to 3.
This provides $\sqrt{16 + (3.3)^2} \cong 5.18$ (as an approximate value) whereas the true value is 5.18556. This is a relative error of -0.00107221 or about 0.1%.

(b) Consider the integral $\displaystyle\int_0^1 \cos(2x^2)\,dx$. Find the trapezoidal rule approximations T_1, T_2, T_4 and T_8 for the integral (using 1, 2, 4 and 8 subdivisions) via an efficient algorithm. From that deduce the Simpsons rule approximations S_2, S_4 and S_8 with 2, 4 and 8 subdivisions using $S_n = (4/3)T_n - (1/3)T_{n/2}$.
Answer. $T_1 = 0.291927$, $T_2 = 0.584755$, $T_4 = 0.648221$, $T_8 = 0.662834$.
$S_2 = 0.682364$, $S_4 = 0.662834$, and $S_8 = 0.667705$.

10. **(a)** A loaded die has the following probabilities of showing the numbers 1-6 after a throw:

$$\frac{2}{15}, \frac{3}{15}, \frac{2}{15}, \frac{3}{15}, \frac{1}{15}, \frac{4}{15}$$

(in that order). Find the probability that an even number will show after the die is thrown.
Answer. $\frac{3}{15} + \frac{3}{15} + \frac{4}{15} = \frac{2}{3}$
A random variable X associated with the experiment of rolling the die has the values

$$X(1) = -5,\ X(2) = 2,\ X(3) = 6,\ X(4) = -2,\ X(5) = 5,\ X(6) = 2.$$

Find the mean and the variance for this random variable.
Answer. Mean is 1. Variance is $88/15$

(b) A factory produces bottles of a soft drink that are sold as 2 litre bottles. A good model is that the quantity of sauce in a bottle obeys a normal distribution with mean 2.03 (litres) and standard deviation 0.12. What proportion of the bottles have less than 2 litres in them?

Answer. 0.4013 (or 40.13%)

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