Linear Algebra 2S2

PRACTICE EXAMINATION SOLUTIONS

1. Find a basis for the row space, the column space, and the nullspace of the following matrix A. Find rank A and nullity A. Verify that every vector in the row space of A is orthogonal to every vector in the nullspace of A.

$$A = \begin{bmatrix} 1 & -1 & 7 & 3 & 4 \\ 1 & -1 & 2 & 3 & 1 \\ -2 & 2 & 1 & -6 & 1 \\ 0 & 4 & 16 & 0 & 8 \end{bmatrix}.$$

Reduce A, getting the 4×5 matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -3/5 \\ 0 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The row space has as basis the three non-zero rows of the reduced matrix, i.e. $\{(1,0,0,3,-3/5), (0,1,0,0,-2/5), (0,0,1,0,3/5)\}$. The column space has basis the first three columns of the original matrix A, since the "first one's" of the reduced matrix appear in these columns: $\{(1,1,-2,0), (-1,-1,2,4), (7,2,1,16)\}$. For the nullspace, using the reduced matrix, we see that x_4 and x_5 are arbitrary, and that $x_1 = -3x_4 + 3/5\ x_5, x_2 = 2/5\ x_5$, and $x_3 = -3/5 x_5$. Thus, the nullspace consists of all vectors in $\mathbb{R}^5$ of the form $(-3x_4 + 3/5\ x_5, 2/5\ x_5, -3/5\ x_5, x_4, x_5)$, which we write as all vectors of the form $x_4(-3,0,0,1,0) + x_5(3/5, 2/5, -3/5, 0, 1)$. In other words, a basis for the nullspace is $\{(-3,0,0,1,0), (3/5,2/5,-3/5,0,1)\}$. The rank of A is 3 and the nullity of A is 2; note that Rank $A+$ Nullity $A = 5$. Finally, to verify that every vector in the row space is orthogonal to every vector in the nullspace, it is enough to check the basis vectors. So, there are 6 things to check, since there are 3 vectors in the basis of the row space and 2 vectors in the basis for the nullspace. Checking the first one: $(1,0,0,3,-3/5) \cdot (-3,0,0,1,0) = 0$. The other 5 verifications are left to you.

2.(a). Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Find an orthogonal matrix P which diagonalizes A. Using this or otherwise, calculate A^8.

(b). Let A be a symmetric matrix, and let λ and μ be two, distinct *eigenvalues of A. Let x be an eigenvector of A corresponding to λ and let y be an eigenvector of A corresponding to μ. Prove that $x \perp y$.*

(a). **The first step** is to find the eigenvalues of A. So, solving $\det(A - \lambda I) =$

$$\det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = 0,$$

we get three roots $\lambda = 1, \lambda = 1, \lambda = 4$ (i.e. 1 is a *double* root).

The second step is to find the corresponding eigenvectors. Our goal is to find *two perpendicular eigenvectors, each of length* 1*, corresponding to* $\lambda = 1$*, and a third eigenvector, of length* 1*, corresponding to* $\lambda = 4$ *which is to be perpendicular to the first two.* For $\lambda = 1$, we therefore consider $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, obtaining the reduced form $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which yields the general solution $(-x_2 - x_3, x_2, x_3)$, where x_2 and x_3 are arbitrary. Letting $x_2 = 1$ and $x_3 = 0$, we get $u_1 = (-1, 1, 0)$ as an eigenvector. Then, letting $x_2 = 0$ and $x_3 = 1$, we get $u_2 = (-1, 0, 1)$ as another eigenvector. Note however, that neither u_1 nor u_2 has length 1; moreover, u_1 is *not* orthogonal to u_2. So, we must apply the Gram-Schmidt process:

Let $v_1 = u_1/||u_1||$, obtaining $v_1 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$. Then, take

$$v_2 = \frac{u_2 - < u_2, v_1 > v_1}{||u_2 - < u_2, v_1 > v_1||},$$

obtaining $v_2 = (-1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$. (Verify that both v_1 and v_2 are unit vectors, that they are both eigenvectors corresponding to $\lambda = 1$, and finally that $v_1 \perp v_2$.)

Now, we find the third eigenvector corresponding to $\lambda = 4$. So, to reduce the matrix $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$. We obtain the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Solving, we get x_3 is arbitrary, $x_1 = x_2 = x_3$. Thus, for example, $u_3 = (1, 1, 1)$ is an eigenvector. However, we need to normalise u_3, i.e. make it have length 1. So, the vector we seek is $u_3/||u_3|| = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Therefore the orthogonal matrix we seek is

$$P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

The point of all of this: For this P, we have

$$P^T A P = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Thus, $A = PDP^T$, so that $A^8 = PD^8P^T = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^8 \end{pmatrix} P^T$.

(b). $\lambda < x, y > = < \lambda x, y > = < Ax, y > = < x, A^T y > = < x, Ay >$, since A is symmetric, $= < x, \mu y > = \mu < x, y >$. Therefore, $(\lambda - \mu) < x, y > = 0$. Now, we are given that $\lambda \neq \mu$. Thus, $< x, y >$ must be 0, i.e. $x \perp y$.

3. In each case, either *explain why the set S in question is a vector space* and *find a basis and dimension of S*, or *explain why S is not a vector space.*

(a). $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ 2x_1 + x_3 - x_4 = 0$ and $x_1 - x_3 - 2x_4 = 0\}$.

(b). $S =$ all 4×4 matrices which are anti-symmetric. *(Recall that A is anti-symmetric means that $A^T = -A$.)*

(c). $S =$ all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 1$.

Part(a). Since S is a subset of the vector space $\mathbb{R}^4$, we need only show that if (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are both in S and c is a scalar then their sum $(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)$ and the scalar product (cx_1, cx_2, cx_3, cx_4) are both in S as well. For example, to verify the second condition, $2(cx_1) + (cx_3) - (cx_4) = c(2x_1 + x_3 - x_4) = c \cdot 0 = 0$ and $(cx_1) - (cx_3) - 2(cx_4) = c(x_1 - x_3 - 2x_4) = c \cdot 0 = 0$. The verification of the first condition is just as simple.

$$S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ \begin{bmatrix} 2 & 0 & 1 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reducing, we get $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, from which the solution $(x_4, x_2, -x_4, x_4) = x_2(0, 1, 0, 0) + x_4(1, 0, -1, 1)$ is obtained. Thus, a basis for S consists of the two vectors $(0, 1, 0, 0), (1, 0, -1, 1)$. $\dim S = 2$.

Part (b). We argue as in part (a). Here, S is a subset of the vector space of all 4×4 matrices, and so we must only verify that if A and B are anti-symmetric and c is a scalar, then $A+B$ and cA are also anti-symmetric. For example, $(A + B)^T = A^T + B^T = (-A) + (-B) = -(A + B)$. Similarly, one shows that $(cA)^T = -(cA)$.

Any 4×4 anti-symmetric matrix A is of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix},$$

which we can write as a sum of 6 matrices

$$A = a_{12} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + ... + a_{34} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Since it is easy to see that these six matrices are linearly independent, they form a basis for S, whose dimension is 6.

Part (c). Note that if f and g are in S, then $f'(0) = g'(0) = 1$. However, $(f+g)'(0) = 2 \neq 1$. Hence S is not a vector space.

4. Let V be a vector space, and let $\{v_1, ..., v_k\} \subset V$.
(a). Define the term $\{v_1, ..., v_k\}$ is linearly independent*.*
(b). Prove: If $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent set, then so is $\{v_1, v_2, v_3\}$.
(c). Find all $k \in \mathbb{R}$ such that the set $\{(1, 2, 3), (0, -2, 1), (1, k^2, 3k - 1)\}$ is linearly independent. Interpret geometrically.

Part (a). The set of vectors $\{v_1, ..., v_k\}$ is *linearly independent* means that the only possible solution $c_1, ..., c_k$ to the equation $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$ is $c_1 = 0, c_2 = 0, ..., c_k = 0$.

Part (b). Suppose that $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent set, and let us show that the subset $\{v_1, v_2, v_3\}$ is also linearly independent. So, consider the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$. If $\{v_1, v_2, v_3\}$ were not linearly independent (i.e. if it were linearly dependent) then there would be a solution where not all the c_1, c_2, c_3 were 0. Then, if we were to set $c_4 = 0$ and $c_5 = 0$, there would be a solution to the equation $c_1v_1 + ... + c_5v_5 = 0$ where not all the $c's$ are zero. But, this contradicts our assumption that $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent.

Part (c). The set of $k \in \mathbb{R}$ such that the set $\{(1, 2, 3), (0, -2, 1), (1, k^2, 3k - 1)\}$ is linearly independent is the same as the set of $k \in \mathbb{R}$ such that $\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & k^2 & 3k-1 \end{pmatrix} \neq 0$. Solving $k^2 + 6k - 10 = 0$, we get $k = -3 \pm \sqrt{19}$. Thus, for *all other* k, $\{(1,2,3), (0,-2,1), (1,k^2,3k-1)\}$ is linearly independent. The geometric interpretation is that if $k = -3 \pm \sqrt{19}$, then the three vectors $\{(1,2,3), (0,-2,1), (1,k^2,3k-1)\}$ lie on the same plane (and hence they are not linearly independent). For all other values of k, $\{(1,2,3), (0,-2,1), (1,k^2,3k-1)\}$ are not co-planar.

5. *In each case,* either *diagonalise the matrix* or *explain why the matrix cannot be diagonalised.*

(a). $A = \begin{pmatrix} -3 & 2 \\ 0 & -3 \end{pmatrix}$. *(b).* $A = \begin{pmatrix} 9 & -9 & 0 \\ 8 & -8 & 0 \\ -14 & 14 & 0 \end{pmatrix}$.

Part (a). It is easy to see that the only eigenvalue of A is the *double root* $\lambda = -3$. Now, reducing the matrix $\begin{pmatrix} -3-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix}$ with $\lambda = -3$, we get the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, we get as eigenvector any vector of the form $(x_1, 0)$. However, there is not a set of *two* linearly independent eigenvectors corresponding to this double root. Hence, the matrix is not diagonalisable.

Part (b). First, the eigenvalues of A are obtained, as usual, by solving $\det(A - \lambda I) = 0$. That is, one must solve the cubic $\lambda^2(\lambda - 1) = 0$.
Second, let's find two-if possible-linearly independent eigenvectors corresponding to the double root $\lambda = 0$: We get x_2 and x_3 are arbitrary, and $x_1 = x_2$. Thus, there are indeed 2 linearly indendent eigenvectors, $(1,1,0), (0,0,1)$. Corresponding to $\lambda = 1$, we find a third eigenvector $(-9,-8,14)$. Thus, if we take $P = \begin{pmatrix} 1 & 0 & -9 \\ 1 & 0 & -8 \\ 0 & 1 & 14 \end{pmatrix}$, then $P^{-1}AP$ will be the diagonal matrix $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

6. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as rotation by an angle θ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the $x-$axis.

(a). Find the standard matrices corresponding to S and T.

(b). Find the real eigenvalues, if any, of S and of T. Interpret your answers geometrically.

(c). Determine whether $S \circ T = T \circ S$.

Part(a). $S \leftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and $T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Part(b). S has no real eigenvalues, unless θ is a multiple of π. The geometric reason is that a rotation will not take a vector into a multiple of itself (unless the rotation is through a very special angle of 0 or $\pm\pi$ or $\pm 2\pi$ or ...).

T has eigenvalues 0 and 1, with corresponding eigenvectors $(1, 0)$ and $(0, 1)$, respectively. Note that T takes $(1, 0)$ to 1 times itself, while T takes $(0, 1)$ to 0 times itself.

Part (c). $S \circ T$ is (almost) never equal to $T \circ S$. One way to see this is to simply multiply the two matrices $S \cdot T$ and $T \cdot S$, and verify that the resulting products are indeed different. Another way is to realise that, geometrically, $T \circ S(x_1, x_2)$ is always a vector on the $x-$axis. On the other hand, $S \circ T(x_1, x_2)$ does not lie on the $x-$ axis (unless the angle of rotation θ is a multiple of π. can be any vector in $\mathbb{R}^2$. So the two compositions, $S \circ T$ and $T \circ S$, are different.