Spectral theory in several variables  Robin Harte

1. Recall the spectrum of a bounded linear operator \( T \in B(X) \) on a Banach space \( X \), more generally a Banach algebra element \( a \in A \):

\[
\sigma(a) = \sigma_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \},
\]

where of course

\[
a \in A^{-1} \iff \exists a', a'' \in A, a'a = 1 = aa'' : \]

for example for square matrices \( A = \mathbb{C}^{n \times n} \) we have the well worn cliche

\[
\sigma(a) = \sigma_A(a) = \{ \lambda \in \mathbb{C} : \det(a - \lambda) = 0 \};
\]

for continuous functions \( A = C(\Omega) \) on compact Hausdorff \( \Omega \) we have the much more revealing and elementary

\[
\sigma(a) = \sigma_A(a) = \{ a(t) : t \in \Omega \}.\]

In a sense spectral theory attempts to reduce general \( a \in A \) to an appropriate \( a^\wedge \in C(\Omega) \). The basic properties of \( \omega = \sigma \) are

\[
\lambda \in \mathbb{C} \implies \omega(\lambda) = \{ \lambda \} \subseteq \mathbb{C}; \]

\[
\omega(a) \text{ is closed and bounded}; \]

\[
\omega(a) \neq \emptyset; \]

\[
f \omega(a) \subseteq f \omega(a) \text{ if } f \in \text{Poly} \]

polynomials in one complex variable);  

\[
\omega f(a) \subseteq f \omega(a) \text{ if } f \in \text{Poly}.\]

There is also the holomorphic functional calculus:

\[
f(a) = \frac{1}{2\pi i} \oint_{\sigma(a)} f(z)(z - a)^{-1} \, dz \text{ if } f \in \text{Holo} \, \sigma(a).\]

Here (1.5) is obvious, (1.6) is the power series for \((1 - z)^{-1}\) and (1.8) is the remainder theorem; more seriously (1.7) and (1.9) require Liouville’s theorem from complex analysis - cf the fundamental theorem of algebra for (1.3). Variants of the spectrum which satisfy most if not all the conditions (1.5)-(1.9) include the left and the right spectrum \( \sigma_{\text{left}} \) and \( \sigma_{\text{right}} \), the left and the right approximate point spectrum \( \tau_{\text{left}} \) and \( \tau_{\text{right}} \), the left and the right point spectrum \( \pi_{\text{left}} \) and \( \pi_{\text{right}} \), the exponential spectrum \( \varepsilon \), the topological boundary \( \partial \sigma \) and the connected hull \( \eta \sigma \).
2. Looking for a multi dimensional analogue, suggest ([7];[8];[11] Definition 11.1.1)

2.1

\[ \sigma(a) = \sigma_{\text{left}}(a) \cup \sigma_{\text{right}}(a) \]

where

2.2

\[ \sigma_{\text{left}}(a) = \{ \lambda \in \mathbb{C}^n : 1 \not\in \sum_{j=1}^n A(a_j - \lambda_j) \} \]

and

2.3

\[ \sigma_{\text{right}}(a) = \{ \lambda \in \mathbb{C}^n : 1 \not\in \sum_{j=1}^n (a_j - \lambda_j)A \} \]

so that of course

2.4

\[ \sigma^2_{\text{right}}(a) = \sigma^B_{\text{left}}(a) \]

where B is obtained from A by “reversal of products”.

It is now rather easy to see that (2.5) (1.6) and (1.8) extend to n-tuples, and that if \( n = 1 \) then \( \sigma(a) \) means what it did before. Unfortunately neither (1.7) nor (1.9) extend ([7];[8];[11] (11.2.3.4)):

2.5

\[ a = (a_1, a_2) = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \implies \sigma(a) = \emptyset. \]

To see this look at some of the consequences of (1.8) ([7];[11] Theorem 11.2.5):

2.6

\[ \omega(a, b) \subseteq \omega(a) \times \omega(b); \]

2.7

\[ f = g \text{ with } h \circ g = z = g \circ h \implies \omega f(a) \subseteq f \omega(a); \]

2.8

\[ f = (z, g) \implies \omega f(a) \subseteq f \omega(a). \]

Indeed for (2.6) apply (1.8) with \( f(z, w) = z \) and with \( f(z, w) = w \), for (2.7) with \( f = g \) and \( f = h \), and for (2.8) with \( f(z, w) = w - g(z) \). Now apply (2.6) to (2.5) to see that \( \sigma(a_1, a_2) \subseteq \{(0,0)\} \): but

\[ a_2a_1 + a_1a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

If we have a “spectrum” \( \omega : a \mapsto \omega(a) \) mapping \( a \in A^n \) to \( \omega(a) \subseteq \mathbb{C}^n \) which is subject to (1.8) then necessary and sufficient for (1.9) will be

2.9

\[ \omega(b) = \omega_{\omega(a)}(b), \]

where

2.10

\[ \omega_{\omega(a)}(b) = \{ \mu \in \mathbb{C}^m : \exists \lambda \in \mathbb{C}^n : (\lambda, \mu) \in \omega(a, b) \} ; \]

for example ([7];[11] (11.2.6.3),(11.2.6.4))

2.11

\[ \omega_{\omega(a)}f(a) = f \omega(a) \]

and

2.12

\[ b = f(a) \implies \omega_{\omega(b)}(a) = \omega(a). \]

To show that, in spite of (2.5), (1.9) and hence (1.7) hold for commuting systems of elements, it is necessary and sufficient to establish (2.9) when \( a \in A^n \) is commutative and commutes with \( b \in A^m \), hence for an induction to establish (2.9) with \( n = 1 \) and \( a = a_1 \) commuting with \( b \): the argument is a version of the Gelfand-Mazur theorem for Banach algebras which says that the “maximal ideal space” of a commutative Banach algebra is big enough to test for invertibility. The detail ([7];[8];[11] Theorem 11.4.3) involves looking at the topological boundary \( \partial \sigma(a + N) \) for the coset of the element \( a \) in an algebra \( M/N \) built from a closed ideal generated by the \( m \)-tuple \( b - \mu \): the idea is extracted from Bunce [1], who proved it for operators on Hilbert space, using states of \( C^* \)-algebras.
3. For a simple application of the spectral mapping theorem in two variables to ordinary spectral theory suppose that \( a \in A \) and that \( p \in A \) satisfies

\[ p = p^2 \text{ and } ap = pa \]

is a commuting projection: then Koliha ([15] Theorem 1.2;[14]) has noticed essentially that if \( 0 \neq \lambda \in \mathbb{C} \) then

\[ a + \lambda(1-p) \in A^{-1} \]

if and only if

\[ 0 \notin \sigma_{pAp}(ap) \text{ and } \sigma_{(1-p)A(1-p)}(a - ap) \subseteq \{0\}, \]

or equivalently

\[ (0,1) \notin \sigma(a,p) \text{ and } ((\alpha,0) \in \sigma(a,p) \implies \alpha = 0). \]

To see this use the spectral mapping theorem to write

\[ \sigma(a + \lambda(1-p)) = \{\alpha : (\alpha,1) \in \sigma(a,p)\} \cup \{\alpha : (\alpha + \lambda,0) \in \sigma(a,p)\}. \]

A similar exercise shows that if \( K \subseteq \mathbb{C} \) with compact \( \sigma(a) \cap K \) and \( \sigma(a) \setminus K \) then \( p \in A \) is determined uniquely by the conditions (3.1) and

\[ \sigma_{pAp}(ap) = \sigma(a) \cap K \text{ and } \sigma_{(1-p)A(1-p)}(a - ap) = \sigma(a) \setminus K, \]

equivalently

\[ \sigma(a,p) = ((\sigma(a) \cap K) \times \{1\}) \cup ((\sigma(a) \setminus K) \times \{0\}). \]

To see this we need to extend the spectral mapping theorem from polynomials to holomorphic functions and then argue, using the characteristic function \( \chi_K \) and the calculus (1.10),

\[ \sigma(p - pq) = \sigma(pq - q) = \{0\} \text{ with } q = \chi_K(a). \]

If (1.8) and (1.9) both hold for systems then Müller ([20] Theorem 2.1) has noticed that this shows up for single elements: whenever \( a, b \in A \) commute

\[ 0 \in \omega(ab) \iff 0 \in \omega(a) \cup \omega(b). \]

In one direction

\[ 0 \in \omega(a) \implies (0,\mu) \in \omega(a,b) \implies 0 = 0\mu \in \omega(ab), \]

and similarly if \( 0 \in \omega(b) \); conversely

\[ \omega(ab) \subseteq \{\lambda\mu : (\lambda,\mu) \in \omega(a) \times \omega(b)\}. \]

There is a natural extension from \( n \)-tuples \( a \in A^n \) of Banach algebra elements to arbitrary systems \( a = (a_x)_{x \in X} \in A^X \), replacing the set \( \{1,2,\ldots,n\} \) by an arbitrary set \( X \): we declare ([9];[11] Definition 11.4.1) \( \lambda \in \mathbb{C}^X \) to be in \( \sigma_{\text{left}}(a) \) iff \( 1 \notin \sum_{x \in X} A(a_x - \lambda_x) \), where the algebraic sum of the left ideals generated by the elements \( a_x - \lambda_x \) is of course the set of all finite sums of left \( A \)-multiples; equivalently \( \lambda \) is in the left spectrum of \( a \) iff every finite restriction \( (\lambda_x)_{x \in Y} \) is in the left spectrum of the corresponding restriction \( (a_x)_{x \in Y} \). The spectral mapping theorem for commuting systems extends to this infinite setting, using Zorn’s lemma instead of mathematical induction; but now it is possible for the index set \( X \) to carry additional structure. From the one way spectral mapping theorem it follows that whenever \( a \in A^X \) is either bounded or continuous or linear or homomorphic then ([9];[11] Theorem 11.4.3) the same is true for every element \( \lambda \in \mathbb{C}^X \) of either the left or the right spectrum of \( a \). In particular this applies when \( X = A \) with \( a_x = x \) for each \( x \in X \): this says that the left and the right spectrum of the algebra \( A \) itself, interpreted as a system of its own elements, reproduces the Gelfand “maximal ideal space”. Thus our spectral mapping theorem not only resembles the Gelfand theorem, it includes it.
4. We can capture analogues of eigenvalues and approximate eigenvalues in several variables; first specialize $A = B(X)$ to operators and then embed $A$ in $B(X)$ with $X = A$: if $T \in B(X)^n$ set

$$\pi^{\text{left}}(T) = \{ \lambda \in \mathbb{C}^n : \bigcap_{j=1}^n (T_j - \lambda_j I)^{-1}(0) \neq \{0\} \}$$

and

$$\tau^{\text{left}}(T) = \{ \lambda \in \mathbb{C}^n : \inf_{\|x\| \geq 1} \sum_{j=1}^n \| (T_j - \lambda_j I)x \| = 0 \},$$

and then, with $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ on $X = A$,

$$\pi^A_{\text{left}}(a) = \pi^A_{\text{left}}(L_a) \text{ and } \pi^A_{\text{right}}(a) = \pi^A_{\text{left}}(R_a)$$

and

$$\tau^A_{\text{left}}(a) = \tau^A_{\text{left}}(L_a) \text{ and } \tau^A_{\text{right}}(a) = \tau^A_{\text{left}}(R_a),$$

We must prove that this is consistent, and can now prove

$$\pi^{\text{right}}(T) = \{ \lambda \in \mathbb{C}^n : \text{cl} \sum_{j=1}^n (T_j - \lambda_j I)X \neq X \}$$

and

$$\tau^{\text{right}}(T) = \{ \lambda \in \mathbb{C}^n : \sum_{j=1}^n (T_j - \lambda_j I)X \neq X \};$$

also

$$\pi^{\text{right}}(L_a) = \tau^{\text{right}}(L_a) = \sigma^A_{\text{right}}(a)$$

and

$$\pi^{\text{right}}(R_a) = \tau^{\text{right}}(R_a) = \sigma^A_{\text{left}}(a).$$

Properties (1.5) and (1.9) hold for (4.3) and (4.4), and (1.6) and the commutative version of (1.7) and (1.9) for (4.4): we prove

$$\pi^{\text{left}}(S) \subseteq \tau^{\text{left}}_{T=1}(S) \text{ if } T = T_1 \text{ commutes with } S \in B(X)^m,$$

and

$$\tau^{\text{left}}Q(T) \subseteq \tau^{\text{left}}(T) \subseteq \pi^{\text{left}}Q(T),$$

where $Q$ is an “enlargement” functor. The argument here is due to Choi, Davis and Rosenthal [3],[6]; for an alternative argument see Slodkowski and Zelazko [22].
5. For operators on a Banach space the “joint spectrum” we have developed seems to be missing something: in particular (Curto [4]) it will not support a functional calculus. For a sense of what is lacking look at a commuting pair $T = (T_1, T_2) \in B(X)^2$ and form (reading from right to left) the Koszul complex ([23];[11] Definition 11.9.1):

$$\begin{pmatrix} 0, (-T_2 & T_1) \\ \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} & 0 \end{pmatrix}.$$  

Three conditions are needed to make this “exact”:

5.1. For operators on a Banach space the “joint spectrum” we have developed seems to be missing something: in particular (Curto [4]) it will not support a functional calculus. For a sense of what is lacking look at a commuting pair $T = (T_1, T_2) \in B(X)^2$ and form (reading from right to left) the Koszul complex ([23];[11] Definition 11.9.1):

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Three conditions are needed to make this “exact”:

5.2.\quad T_1^{-1}(0) \cap T_2^{-1}(0) = \{0\};

5.3.\quad T_1(X) + T_2(X) = X;

5.4.\quad (-T_2 & T_1) \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}^{-1}(0) \subseteq \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}(X).

Now the “Taylor spectrum” of $T$ is the set of $\lambda \in \mathbb{C}^2$ for which the Koszul complex of $T - \lambda I$ is exact, and the “Taylor split spectrum” for which it is “splitting exact”, in the sense that there are $T'$ and $T''$ for which

$$\begin{pmatrix} T'_1 & T'_2 \\ T'_2 & T_2 \end{pmatrix} (T_1, T_2) = I = \begin{pmatrix} -T_2 & T_1 \\ T_1 & T'_1 \end{pmatrix} \begin{pmatrix} -T_2' & T_1' \\ T_1' & T''_1 \end{pmatrix}$$

and

$$\begin{pmatrix} T_1' \\ T_2' \end{pmatrix} (T_1, T_2) + \begin{pmatrix} -T_2'' & T_1'' \\ T_1'' & T''_1 \end{pmatrix} (-T_2, T_1) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$  

At first sight it seems quite difficult [10] to tell them apart. To prove the spectral mapping theorem for the Taylor (split) spectrum ([23];[11] Theorem 11.9.10), first piling the complex (5.1) into a single matrix [10] and then doing induction on its length, we argue ([11] Theorem 10.9.5)

$$\begin{pmatrix} R, S \) \text{ and } (S, T) \text{ (split)exact } \implies \begin{pmatrix} R \\ V \end{pmatrix}, \begin{pmatrix} S \\ U \end{pmatrix} \text{ (split)exact}$$

provided $VS = SU$, and conversely ([11] Theorem 10.9.6)

$$\begin{pmatrix} (R, S) \end{pmatrix}, \begin{pmatrix} S \\ U \end{pmatrix} \text{ (split)exact } \implies \begin{pmatrix} R \\ V \end{pmatrix}, \begin{pmatrix} S \text{ and } (S, (T \cdot U)) \text{ (split)exact.}$$

It is an entertaining, if somewhat meaningless, observation that if $T = (T_1, T_2) \in B(X)^2$ with $X = \mathbb{C}^2$ is given by the pair of matrices of (2.5) then the Taylor spectrum of $T$ is not empty:

$$\begin{pmatrix} -T_2 & T_1 \end{pmatrix}^{-1}(0) \cap \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}(X) = \{0\}.$$  

Thus the Taylor spectrum of $T$ is $\{0\}$. Of course the spectral mapping theorem fails, both ways: neither (1.8) nor (1.9) hold with $f = z_1z_2 + z_2z_1$.

Müller [20] has noticed that (3.9) fails when $\omega$ is derived from either “Kato invertibility” [12] or “Kato non singularity” [13]. We shall call an operator $T \in B(X)$ hyperexact if it satisfies the “Saphar condition”

$$T^{-1}(0) \subseteq T^\infty(X) = \bigcap_{n=1}^\infty T^n(X),$$

equivalently

$$\bigcup_{n=1}^\infty T^{-n}(0) = T^{-\infty}(0) \subseteq T(X).$$

Now $T \in B(X)$ will be called Kato non singular if it is hyperexact with closed range,

$$T(X) = cl T(X),$$

and will be called Kato invertible if it is hyperexact with a generalized inverse, $T^\dagger \in B(X)$ for which

$$T = TT^\dagger T.$$  

(5.10) and (5.13) combine to give $T - zI : U \to B(X)$ a “holomorphic” generalized inverse $T - zI)^\dagger$ on some neighbourhood $U$ of $0 \in \mathbb{C}$. It turns out that in both cases the induced $\omega$ satisfies conditions (1.1)-(1.5); the arguments are extracted from Mbekhta [18],[19] for operators on Hilbert space, where of course both conditions are the same. Müller’s example ([20] Example 2.2) to violate (3.9) is on Hilbert space; the operators are in a sense mutually orthogonal shifts:
6. Example There are commuting operators $S$ and $T$ for which $S$ and $T$ are Kato invertible while $ST$ is not.
For take, with

6.1 \[ X = \ell_2 = \text{cl} \sum_{ij \leq 0} C e_{i,j}, \]

6.2 \[ Te_{i,j} = 0 \text{ if } i = 0 < j, = e_{i+1,j} \text{ else} \]
and

6.3 \[ Se_{i,j} = 0 \text{ if } j = 0 < i, = e_{i,j+1} \text{ else}. \]
Now claim

6.4 \[ ST = TS; \]

6.5 \[ T^{-1}(0) = \text{cl} \sum_{i=0<j} C e_{i,j} = \text{cl} \sum_{j=1}^{\infty} C e_{0,j}; \]

6.6 \[ S^{-1}(0) = \text{cl} \sum_{j=0<i} C e_{i,j} = \text{cl} \sum_{i=1}^{\infty} C e_{i,0}; \]

6.7 \[ T^{-\infty}(0) = \text{cl} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} C e_{i,j}; \]

6.8 \[ S^{-\infty}(0) = \text{cl} \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} C e_{i,j}; \]

6.9 \[ (ST)^{-1}(0) = \text{cl} \sum_{ij=0} C e_{i,j}; \]

6.10 \[ T(X) = (\sum_{j=-\infty}^{-1} C e_{0,j})^{\perp}; \]

6.11 \[ S(X) = (\sum_{i=-\infty}^{-1} C e_{i,0})^{\perp}; \]

6.12 \[ T^\infty(X) = (\sum_{j=-\infty}^{-1} \sum_{i=0}^{\infty} C e_{i,j})^{\perp}; \]

6.13 \[ S^\infty(X) = (\sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} C e_{i,j})^{\perp}; \]

6.14 \[ (ST)(X) = \text{cl} \sum_{(i,j) \neq (0,0)} C e_{i,j}. \]
Thus

6.15 \[ T^{-\infty}(0) \subseteq T(X) \text{ and } S^{-\infty}(0) \subseteq S(X), \]
while in particular

6.16 \[ e_{0,0} \in (ST)^{-1}(0) \setminus ST(X). \]
In a formally similar example Müller [21] shows that the Taylor spectrum and the Taylor split spectrum need not coincide:

7. Example There are commuting operators $S$, $T$ for which $(S, T)$ is Taylor non-singular but not Taylor invertible.

For take $W = \bigoplus_{i,j} W_{i,j}$ with

7.1 $W_{i,j} = X \ (1 \leq \min(i, j)), = Y_1 \ (j \leq 0 \leq 1 \leq i), = Y_2 \ (i \leq 0 \leq 1 \leq j), = Y_1 \cap Y_2 \ (\max(i, j) \leq 0)$

where $Y_1, Y_2$ are closed subspaces of $x \cdot Z \cdot X$ for which

7.2 $Y_1 + Y_2 = X$ and $\{(x, x) : x \in Y_1 \cap Y_2\}$ is not complemented in $Y_1 \oplus Y_2$.

Evidently

7.3 $W_{i,j} = W_{i+1,j} \cap W_{i,j+1}$.

Now set

7.4 $(Tw)_{i,j} = w_{i-1,j}$ and $(Sw)_{i,j} = w_{i,j-1}$:

we claim

7.5 $\begin{pmatrix} T \\ S \end{pmatrix}$ is bounded below

and

7.6 $(-S \ T)$ is onto

and

7.7 $\begin{pmatrix} T \\ S \end{pmatrix} W$ is not complemented in $\begin{pmatrix} W \\ W \end{pmatrix}$.