Reading Feynman Rules from Lagrangean

Robert Clancy

May 16, 2008

I work with $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

1 Free Propagators

1.1 General Scheme

Given a momentum-space representation of a propagator

$$\Delta(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik(x-y)} \tilde{\Delta}(k)$$

the associated factor for a line is

$$\frac{1}{i}\tilde{\Delta}(k)$$

where k is the momentum assigned to the line.

1.1.1 Real fields

$$\mathcal{L} = -\frac{1}{2}\chi^{\mathrm{T}} \mathrm{D} \chi$$

EOM

$$\mathrm{D}\chi = 0$$

Propagator

$$\mathrm{D}\Delta = \delta$$

1.1.2 Complex fields

$$\mathcal{L} = -\chi^{\dagger} \mathrm{D} \chi$$

EOM

$$\mathrm{D}\chi = 0$$
$$\chi^\dagger \overleftarrow{\mathrm{D}^\dagger} = 0$$

Propagator

$$\mathrm{D}\Delta = \delta$$

1.2 Scalar fields

Real

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\ =& -\frac{1}{2}\phi(\partial^2 + m^2)\phi + \partial_\mu(\frac{1}{2}\phi\partial^\mu\phi) \\ =& -\frac{1}{2}\phi(\partial^2 + m^2)\phi\end{aligned}$$

(up to a total divergence)

Complex

$$\mathcal{L} = -\phi^*(\partial^2 + m^2)\phi$$

EOM

$$(\partial^2 + m^2)\phi$$

Propagator

$$(\partial^2 + m^2)\Delta(x - y) = \delta^4(x - y)$$

taking Fourier transform

$$(-k^2 + m^2)\tilde{\Delta}(k) = 1$$
$$\Delta(x - y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{-k^2 + m^2 - i\epsilon}$$

1.3 Dirac Spinor field

$$\mathcal{L} = -\bar{\psi}(-i\partial\!\!\!/ + m)\psi$$

EOM

$$(-i\partial\!\!\!/ + m)\psi = 0$$
$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0$$

Propagator

$$(-i\partial\!\!\!/ + m)S(x-y) = \delta^4(x-y)$$

taking Fourier transform

$$(-p\!\!\!/ + m)\tilde{S}(k) = 1$$
$$S(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{p\!\!\!/ + m}{-p^2 + m^2 - i\epsilon} e^{-ip(x-y)}$$

1.4 U(1) Gauge field? (photon)

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) \\ &= -\frac{1}{2}(-A^\nu \partial^\mu \partial_\mu A_\nu + A^\nu \partial^\mu \partial_\nu A_\mu) + \partial_\mu K^\mu \\ &= -\frac{1}{2}A^\mu(-g_{\mu\nu}\partial^2 + \partial_\mu \partial_\nu)A^\nu\end{aligned}$$

(up to a total divergence)

EOM

$$(-g_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu)A^\nu = 0$$

(in Lorenz[1] gauge)

$$-g_{\mu\nu}\partial^2 A^\nu = 0$$

[1] note Lorenz gauge is often incorrectly called Lorentz gauge

Propagator

I'm not sure if the following derivation is correct because I have not been careful about gauge etc. but it gives the right answer for Lorenz gauge.

$$-g_{\mu\rho}\partial^2\Delta^{\rho\nu}(x-y) = \delta_\mu{}^\nu\delta^4(x-y)$$

taking Fourier transform

$$-g_{\mu\rho}(-k^2)\tilde{\Delta}^{\rho\nu}(k) = \delta_\mu{}^\nu$$

$$\Delta^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)}$$

2 Vertex Factors

2.1 No derivatives

Consider $\chi_1\dots\chi_s$ distinct (fully contracted) fields (hermitian conjugate fields are considered distinct) with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where $\mathcal{L}_0$ is the free field Lagrangean and

$$\mathcal{L}_I = \lambda\chi_1^{n_1}\dots\chi_s^{n_s}$$

then the associated vertex factor is

$$i(n_1!)\dots(n_s!)\lambda$$

By fully contracted I mean that if the field carries any indices then these are contracted to yield a scalar e.g. $(\bar{\psi}\psi)^n$ and $(A^\mu A_\mu)^n$, see later examples (definitely not required for course). In these cases one considers pairs of contracted fields.

Examples

2.1.1 ϕ^4

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4$$

vertex factor:

$$-i\lambda$$

2.1.2 Yukawa

$$\mathcal{L}_I = g\phi\bar{\psi}\psi$$

vertex factor:

$$ig$$

2.1.3 QED

$$\mathcal{L}_I = -eA_\mu\bar{\psi}\gamma^\mu\psi$$

vertex factor:

$$-ie\gamma^\mu$$

2.1.4 Crazier ones (not on course)

$$\mathcal{L}_I = \lambda(\phi^*\phi)^2(\bar{\psi}\psi)^3$$

vertex factor:

$$i\lambda(2!2!3!)$$

The fermionic lines must be considered in pairs and one needs a notation for pairing lines in the diagrams, for example drawing a dash across first pair of lines, two dashes on the second pair, three dashes etc., otherwise there would be ambiguity about which line to follow when contracting spinor indices.

$$\begin{aligned}\mathcal{L}_I &= \lambda(\phi^*\phi)(A^\mu A_\mu)^2 \\ &= \lambda(\phi^*\phi)(g_{\mu\nu}A^\mu A^\nu)(g_{\rho\sigma}A^\rho A^\sigma)\end{aligned}$$

vertex factor:

$$i\lambda g_{\mu\nu}g_{\rho\sigma}(2!)$$

Similarly one needs to specify which of the photon lines are paired so one knows how to contract the indices.

2.2 Derivatives

Not so sure about this but it works for the only example I've looked at so thats 100% success rate.

$$\begin{aligned}\partial_\mu\chi &\rightarrow -ik_\mu \\ \partial_\mu\chi^\dagger &\rightarrow +ik_\mu\end{aligned}$$

SED

$$\mathcal{L} = -\frac{1}{2}A^\mu(-g_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu)A^\nu + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

$$\mathcal{L}_I = ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2 g_{\mu\nu}A^\mu A^\nu\phi^*\phi$$
$$= \mathcal{L}_1 + \mathcal{L}_2$$

vertex factor associated with $\mathcal{L}_1$:

$$-ie(k' + k)_\mu$$

vertex factor associated with $\mathcal{L}_2$:

$$2ie^2 g_{\mu\nu}$$

3 External Lines

External scalar:

$$1$$

Incoming fermion:

$$u_s(p)$$

Outgoing fermion:

$$\bar{u}_s(p)$$

Incoming anti-fermion:

$$\bar{v}_s(p)$$

Outgoing anti-fermion:

$$v_s(p)$$

Incoming photon:

$$\varepsilon_\lambda^{\mu *}(k)$$

Outgoing photon:

$$\varepsilon_\lambda^\mu(k)$$

Required Identities for spin/polarisation averaged Cross-sections:

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$
$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu \varepsilon_\lambda^{\nu *} \rightarrow g^{\mu\nu}$$

the last identity is understood to be only used when inside a scattering amplitude.

Also some other facts which follow from definitions of u, v which are useful:

$$
\begin{aligned}
(-\not{p} + m)u_s(p) &= 0 \\
\bar{u}_s(p)(-\not{p} + m) &= 0 \\
(\not{p} + m)v_s(p) &= 0 \\
\bar{v}_s(p)(\not{p} + m) &= 0
\end{aligned}
$$

4 Summary of Rules

1. For incoming particles draw arrows pointing towards the vertex, for outgoing particles draw arrows pointing away from the vertex. For incoming anti-particles draw arrows pointing away from the vertex, for outgoing anti-particles draw arrows pointing towards the vertex. Assign momentum k^μ/p^μ as appropriate to particles and $-k^\mu/-p^\mu$ to anti-particles, conserving momentum at each vertex.

2. Each vertex, propagator and external line is given its appropriate factor as derived above.

3. Contract indices along fermionic/photon lines as one would expect. For a fermionic line start with a barred spinor (at the end of the line) and contract back to the start.

4. Relative signs of diagram containing fermion lines is found by arranging all fermionic lines horizontal and comparing sings of permutations with an extra $-$ sign for fermionic loops.

5. Integrate over loop momenta with measure $\mathrm{d}^4l/(2\pi)^4$.

6. Divide by symmetry factor (considering sources fixed).