

The main results

1. Show that every solution of the wave equation $u_{tt} = c^2 u_{xx}$ has the form

$$u(x, t) = F(x + ct) + G(x - ct).$$

2. Prove d'Alembert's formula for the initial value problem for the wave equation. You may assume that the unique solution has the form $u(x, t) = F(x + ct) + G(x - ct)$.
3. Show that the solution of the heat (or wave) equation on the real line is even (or odd), if the initial data are even (or odd). You may assume the solution is unique.
4. Solve the Dirichlet (or Neumann) problem for the heat (or wave) equation on the half line. You may use the explicit formula for the problem on the whole real line.
5. Prove conservation of energy for solutions of the Dirichlet (or Neumann) problem for the wave equation on $[0, L]$.
6. Prove uniqueness of solutions for the Dirichlet (or Neumann) problem for the wave equation on $[0, L]$. You may assume conservation of energy.
7. Show that the wave equation on the real line is stable with respect to the L^∞ norm on any finite time interval $[0, T]$. You may use d'Alembert's formula.
8. State and prove the maximum principle for the heat equation $u_t = k u_{xx}$.
9. Show that the solution $u(x, t)$ of the heat equation on the real line is such that

$$\int_{-\infty}^{\infty} |u(x, t)|^p dx \leq \int_{-\infty}^{\infty} |u(x, 0)|^p dx$$

for any $p \geq 1$ and $t \geq 0$. You may use the explicit formula for the solution, Hölder's inequality and any of the known facts about the heat kernel.

10. State and prove the mean value property for harmonic functions over spheres.
11. State and prove the maximum principle for the Laplace equation. You may use the mean value property for harmonic functions over spheres.
12. Show that every harmonic function is smooth. You may use the mean value property for harmonic functions over spheres.
13. State and prove Green's identities. You may use the divergence theorem.
14. Prove uniqueness of solutions for the Dirichlet problem for the Laplace equation over a bounded region. You may use either the maximum principle or Green's identities.

15. Prove Liouville's theorem for harmonic functions in $\mathbb{R}^n$. You may use the mean value property for harmonic functions over balls, the divergence theorem and the fact that every harmonic function is smooth.
16. Letting $F(x) = \frac{1}{4\pi|x|}$ for all $x \in \mathbb{R}^3$, show that $-\Delta F = \delta$ in the sense of distributions. You may assume that $\Delta F(x) = 0$ at all points $x \neq 0$.
17. State and prove the Dirichlet principle for harmonic functions in $\mathbb{R}^n$.
18. Suppose $u(x, t)$ satisfies the wave equation $u_{tt} = c^2 \Delta u$ in $\mathbb{R}^n$ and let $U(x, t, r)$ be the mean value of u over the sphere of radius r around x . Show that

$$U_{tt} = c^2 \left(\frac{n-1}{r} \cdot U_r + U_{rr} \right).$$

19. Suppose that u is a weak solution of $u_t + F(u)_x = 0$ which is discontinuous across the curve $x = h(t)$ but smooth on either side of the curve. Show that

$$h'(t) = \frac{F(u^+) - F(u^-)}{u^+ - u^-},$$

where u^- and u^+ are the limits of $u(x, t)$ as x approaches $h(t)$ from the left and the right, respectively. You may use the divergence theorem.

Some general comments

- For the first section of the exam (MA3425), you will need to answer 3/4 questions based on homework assignments 1, 2 and the results 1–8 above.
- For the second section of the exam (MA3426), you will need to answer 3/4 questions based on homework assignments 3, 4, 5 and the results 9–19 above.
- You should not provide definitions or statements of results, unless you are asked to. There is a strict marking scheme associated with each sophister exam.
- A sample question would be the following.
 - (a) [4 points] Show that the solution $u(x, t)$ of the wave equation on the real line is even in x , if the initial data are even. You may assume the solution is unique.
 - (b) [6 points] Solve the Neumann problem for the wave equation on the half line. You may use d'Alembert's formula.
 - (c) [5 points] Show that the solution $u(x, t)$ of the wave equation on the real line is bounded, if $|u(x, 0)| \leq 1$ and $|u_t(x, 0)| \leq 1/(1 + x^2)$ for all $x \in \mathbb{R}$.
 - (d) [5 points] Find the solution of the non-homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = t, \quad u(x, 0) = 1, \quad u_t(x, 0) = x.$$