Analysis Problems #9
Solutions

1. *An ant starts out at the origin in the xy-plane and walks* 1 *unit south, then* $1/2$ *units east, then* $1/4$ *units north, then* $1/8$ *units west, then* $1/16$ *units south, and so on. If it continues like that indefinitely, which point in the xy-plane will it eventually reach?*

- When it comes to the vertical direction, the ant moves by a total of

$$y = -1 + \frac{1}{4} - \frac{1}{16} + \ldots = -\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n$$

units. Using the formula for a geometric series, we then get

$$y = -\frac{1}{1+1/4} = -\frac{4}{5}\,.$$

When it comes to the horizontal direction, it moves by a total of

$$x = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \ldots = \frac{1}{2}\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1/2}{1+1/4} = \frac{2}{5}$$

units. In particular, the ant will eventually reach the point $(2/5, -4/5)$.

2. *Find the Taylor series expansion of $f(x) = \log(1+x)$ around the point $x = 0$. Hint: find the nth derivative of $f(x)$ explicitly.*

- The first few derivatives of $f(x)$ are

$$f'(x) = (1+x)^{-1}, \qquad f''(x) = -(1+x)^{-2}, \qquad f'''(x) = 2(1+x)^{-3}.$$

Based on this fact, we now guess that the nth derivative is

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}, \qquad n \geq 1.$$

When $n = 1$, the last formula holds by above. If it holds for some n, then

$$f^{(n+1)}(x) = (-1)^{n-1}(n-1)!\,(-n)(1+x)^{-n-1} = (-1)^n n!\,(1+x)^{-n-1}$$

so it also holds for $n+1$. Thus, the formula holds for any $n \geq 1$ whatsoever. Once we now note that $f(0) = \log 1 = 0$, we get the Taylor series expansion

$$T(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} \cdot x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n.$$

3. *Find the Taylor series expansion of* $f(x) = \arctan x$ *around the point* $x = 0$*. Hint: you already know the Taylor series expansion of* $f'(x)$*.*

- Using the formula for a geometric series, one easily gets

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrating term by term, we can then obtain the Taylor series for $f(x)$, namely

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \,.$$

4. *Test the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{e^{1/n} - 1}{n} \,.$$

- In this case, we use the limit comparison test with

$$a_n = \frac{e^{1/n} - 1}{n} \,, \qquad b_n = \frac{1}{n^2} \,.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{1/n}(1/n)'}{(1/n)'} = e^0 = 1$$

by L'Hôpital's rule. Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ must also converge.