Sample exam
Solutions

1. *Show that the set* $A = \{\frac{2n+3}{n+4} : n \in \mathbb{N}\}$ *has a minimum but no maximum.*

- First, note that $\frac{2+3}{1+4} = 1$ is an element of A and that

$$1 \leq \frac{2n+3}{n+4} \iff n+4 \leq 2n+3 \iff 1 \leq n.$$

Since the rightmost inequality is true, the leftmost one is true as well. This makes 1 an element of A which is at least as small as any other element, so $\min A = 1$.

- To show that A has no maximum, suppose $x = \frac{2n+3}{n+4}$ is the maximum and let

$$y = \frac{2(n+1)+3}{(n+1)+4} = \frac{2n+5}{n+5}\,.$$

Then y is an element of A, and it is actually larger than x because

$$\frac{2n+3}{n+4} < \frac{2n+5}{n+5} \iff 2n^2+13n+15 < 2n^2+13n+20$$
$$\iff 15 < 20.$$

As no element of A can be larger than the maximum of A, this is a contradiction.

2. *Suppose that* f *is continuous with* $f(0) < 1$. *Show that there exists some* $\delta > 0$ *such that* $f(x) < 1$ *for all* $-\delta < x < \delta$.

- Since $\varepsilon = 1 - f(0)$ is positive, some $\delta > 0$ exists such that

$$|x-0| < \delta \implies |f(x) - f(0)| < \varepsilon \implies f(x) - f(0) < 1 - f(0).$$

In other words, one has $f(x) < 1$ for all $-\delta < x < \delta$, as needed.

3. *Show that the function* f *defined by*

$$f(x) = \left\{ \begin{array}{ll} 2x+1 & \text{if } x \leq 1 \\ x+3 & \text{if } x > 1 \end{array} \right\}$$

is not continuous at $y = 1$.

- Suppose that f is continuous at $y = 1$. Then there exists some $\delta > 0$ such that

$$|x-1| < \delta \implies |f(x) - f(1)| < 1. \qquad (*)$$

We now examine the last equation for the choice $x = 1 + \frac{\delta}{2}$. On one hand, we have

$$|x-1| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x) - f(1)| = |x + 3 - 3| = 1 + \frac{\delta}{2} > 1$$

because $x = 1 + \frac{\delta}{2} > 1$ here. This actually violates the conclusion in equation $(*)$.

4. *Suppose that f is continuous with $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R}$. Show that f is constant.*

- Suppose not. Then we have $f(a) < f(b)$ for some $a, b \in \mathbb{R}$ and the intermediate value theorem ensures that f attains every value in between. Since an irrational number lies between any two reals, f must also attain irrational values, a contradiction.

5. *Give an example of a bounded function which does not have a maximum on* $[0, 1]$.

- There are obviously many examples. To obtain a typical one, start with any function that has a maximum and change the maximum value to a smaller one; the resulting function will then fail to have a maximum. As a simple example, let

$$f(x) = \left\{ \begin{array}{cc} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{array} \right\}.$$

Then $f(x)$ is bounded on $[0, 1]$ but it does not attain a maximum value on $[0, 1]$.

6. *Let $p > 0$ be fixed. Show that $\frac{\log x}{x^p} \leq \frac{1}{pe}$ for all $x > 0$.*

- Setting $f(x) = x^{-p} \log x$ for convenience, we can use the product rule to get

$$f'(x) = -px^{-p-1} \log x + x^{-p} x^{-1} = x^{-p-1} (1 - p \log x).$$

Since $x > 0$ by assumption, this expression is positive when

$$p \log x < 1 \quad \iff \quad \log x < 1/p \quad \iff \quad x < e^{1/p}$$

and negative when $x > e^{1/p}$. In particular, f is increasing for the former values of x and decreasing for the latter, so f attains its maximum value when $x = e^{1/p}$ and

$$f(x) \leq f(e^{1/p}) = (e^{1/p})^{-p} \log e^{1/p} = \frac{1}{pe}.$$

7. *Suppose that f is decreasing on $[0, 1]$. Show that f is integrable on $[0, 1]$.*

- Let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[0, 1]$ into n subintervals of length $1/n$. Since f is decreasing, we then have

$$\inf_{[x_k, x_{k+1}]} f(x) = f(x_{k+1}) \quad \text{for each } 0 \leq k \leq n - 1$$

and this implies that

$$\begin{aligned} S^-(f,P) &= \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x)\cdot(x_{k+1}-x_k) \\ &= \frac{1}{n}\Big(f(x_1)+f(x_2)+\ldots+f(x_n)\Big). \end{aligned}$$

Using the exact same reasoning to compute the upper Darboux sum, we now get

$$\begin{aligned} S^+(f,P) &= \sum_{k=0}^{n-1} \sup_{[x_k,x_{k+1}]} f(x)\cdot(x_{k+1}-x_k) \\ &= \frac{1}{n}\Big(f(x_0)+f(x_1)+\ldots+f(x_{n-1})\Big), \end{aligned}$$

so we may combine the last two equations to arrive at

$$S^+(f,P) - S^-(f,P) = \frac{f(x_0)-f(x_n)}{n} = \frac{f(0)-f(1)}{n}.$$

Letting n be sufficiently large, we can certainly make this expression as small as we wish, hence f is integrable by Theorem 4.9 in your notes.

8. *Show that there exists a unique function f, defined for all $x \in \mathbb{R}$, such that*

$$f'(x) = e^{-x^2}, \qquad f(0) = 0.$$

Moreover, show that f is odd, namely that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

- Since e^{-x^2} is continuous, the fundamental theorem of calculus ensures that

$$f(x) = \int_0^x e^{-t^2}\,dt$$

has the desired properties. To prove uniqueness, suppose that

$$g'(x) = e^{-x^2}, \qquad g(0) = 0.$$

Then $g'(x) = f'(x)$, so $g(x) = f(x) + C$ and we can let $x = 0$ to get

$$C = g(0) - f(0) = 0 \quad\Longrightarrow\quad g(x) = f(x) + C = f(x).$$

- To show that f is odd, let $h(x) = f(x) + f(-x)$ and compute

$$h'(x) = f'(x) - f'(-x) = e^{-x^2} - e^{-x^2} = 0.$$

Since $h(x)$ is constant, we find $h(x) = h(0) = 0$ and thus $f(-x) = -f(x)$.

9. *Define a sequence* $\{a_n\}$ *by letting* $a_1 = 1$ *and*

$$a_{n+1} = 3 + \sqrt{a_n} \quad \textit{for each } n \geq 1.$$

Show that $1 \leq a_n \leq a_{n+1} \leq 9$ *for each* $n \geq 1$ *and find the limit of this sequence.*

- Since the first two terms are $a_1 = 1$ and $a_2 = 3 + 1 = 4$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 9$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 1 \leq \sqrt{a_n} \leq \sqrt{a_{n+1}} \leq 3 \quad &\implies \quad 4 \leq a_{n+1} \leq a_{n+2} \leq 6 \\ &\implies \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 9. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = 3 + \sqrt{a_n} \quad \implies \quad \lim_{n\to\infty} a_{n+1} = 3 + \lim_{n\to\infty} \sqrt{a_n} \quad \implies \quad L = 3 + \sqrt{L}.$$

To solve this equation, we now set $x = \sqrt{L}$ to get

$$x^2 = 3 + x \quad \implies \quad x^2 - x - 3 = 0 \quad \implies \quad x = \frac{1 \pm \sqrt{13}}{2}.$$

Since $x = \sqrt{L}$ must be non-negative, only one of the roots is acceptable, and so

$$L = x^2 = \left(\frac{1+\sqrt{13}}{2}\right)^2 = \frac{1 + 13 + 2\sqrt{13}}{4} = \frac{7 + \sqrt{13}}{2}.$$

10. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \sin\frac{1}{n}, \qquad \sum_{n=1}^{\infty} \cos\frac{1}{n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n}.$$

- To test the first series for convergence, we use the limit comparison test with

$$a_n = \sin\frac{1}{n}, \qquad b_n = \frac{1}{n}.$$

Note that the limit comparison test is applicable by L'Hôpital's rule, as

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(1/n)(1/n)'}{(1/n)'} = \cos 0 = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

- The second series diverges by the nth term test because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1.$$

- To test the third series for convergence, we use the alternating series test with

$$a_n = \frac{e^{1/n}}{n}.$$

Note that a_n is non-negative for each $n \geq 1$, and that we also have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = \lim_{n\to\infty} \frac{e^0}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Moreover, a_n is decreasing for each $n \geq 1$ because

$$\left(\frac{e^{1/n}}{n}\right)' = \frac{e^{1/n}(-1/n^2)\cdot n - e^{1/n}}{n^2} = -\frac{e^{1/n}(1+n)}{n^3} < 0$$

for such n. Thus, the third series converges by the alternating series test.

11. *Suppose that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges when $x = 2$. Show that it converges absolutely when $|x| < 2$.

- Since $\sum_{n=0}^{\infty} 2^n a_n$ converges, the sequence $2^n a_n$ is convergent, hence also bounded. In particular, there exists some positive constant C_0 such that

$$|2^n a_n| \leq C_0 \quad \text{for all } \; n \geq 0.$$

Using this fact, we now find that

$$\sum_{n=0}^{\infty} |a_n x^n| = \sum_{n=0}^{\infty} |2^n a_n| \cdot \frac{|x|^n}{2^n} \leq C_0 \sum_{n=0}^{\infty} \left(\frac{|x|}{2}\right)^n.$$

If $|x| < 2$, then the series on the right is a convergent geometric series, so the series on the left converges by comparison. This shows that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.