Analysis Homework #8
Solutions

1. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot \log\left(1+\frac{1}{n}\right), \qquad \sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n}.$$

- To test the first series for convergence, we use the limit comparison test with

$$a_n = \frac{1}{n} \cdot \log\left(1+\frac{1}{n}\right), \qquad b_n = \frac{1}{n^2}.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \cdot \log\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \log\left(1+\frac{1}{n}\right)^n = \log e = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also converge.

- To test the second series for convergence, we use the alternating series test with

$$a_n = \frac{e^{1/n}}{n}.$$

Note that a_n is non-negative for each $n \geq 1$, and that we also have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = \lim_{n\to\infty} \frac{e^0}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Moreover, a_n is decreasing for each $n \geq 1$ because

$$\left(\frac{e^{1/n}}{n}\right)' = \frac{e^{1/n}(-1/n^2) \cdot n - e^{1/n}}{n^2} = -\frac{e^{1/n}(1+n)}{n^3} < 0$$

for such n. Thus, the second series converges by the alternating series test.

2. *Find the radius of convergence for each of the following power series:*

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

- When it comes to the first power series, we have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{3^n}{3^{n+1}} = \frac{n+1}{3n} \cdot x$$

and this implies

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{3n} \cdot |x| = \frac{|x|}{3}.$$

In particular, the given series converges when $|x| < 3$ and it diverges when $|x| > 3$, so its radius of convergence is $R = 3$.

- When it comes to the second power series, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n}$$

  and this implies

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2 \cdot |x|}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1) \cdot |x|}{4n+2} = \frac{|x|}{4}\,.$$

  In particular, the given series converges when $|x| < 4$ and it diverges when $|x| > 4$, so its radius of convergence is $R = 4$.

3. *Assuming that $|x| < 1$, use the formula for a geometric series to show that*

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}\,.$$

- Since $|x| < 1$, the formula for a geometric series applies to give

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}.$$

  Differentiating both sides of this equation and multiplying by x, we now get

$$\sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2} \quad \Longrightarrow \quad \sum_{n=0}^{\infty} nx^n = x(1-x)^{-2} = \frac{x}{(1-x)^2}\,.$$

4. *Let $\alpha \in \mathbb{R}$ be fixed. Find the radius of convergence for the binomial series*

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \cdot x^n.$$

- Using the ratio test as usual, we get

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\alpha - n}{n+1} \cdot x \right| = |x|,$$

  so it easily follows that the radius of convergence is $R = 1$.