

**2010 Final exam**  
**Solutions**

1. Show that the set  $A = \{\frac{n+1}{n+2} : n \in \mathbb{N}\}$  is such that  $\sup A = 1$ .

- To show that 1 is an upper bound of  $A$ , we note that

$$\frac{n+1}{n+2} \leq 1 \iff n+1 \leq n+2 \iff 1 \leq 2.$$

Since the rightmost inequality is true, the leftmost one must also be true.

- To show that 1 is the least upper bound, we show that no number  $x < 1$  is an upper bound. Let us then fix some  $x < 1$  and try to find an element of  $A$  which is bigger than  $x$ . Since every element of  $A$  has the form  $\frac{n+1}{n+2}$ , we need to make sure that

$$\begin{aligned} \frac{n+1}{n+2} > x &\iff n+1 > nx+2x \\ &\iff (1-x)n > 2x-1 &\iff n > \frac{2x-1}{1-x}. \end{aligned}$$

According to one of our theorems, we can always find an integer  $n$  that satisfies this inequality. Then, our computation above shows that  $\frac{n+1}{n+2}$  is an element of  $A$  which is bigger than  $x$ . This also means that  $x$  is not an upper bound of  $A$ , as needed.

2. Show that  $f$  is continuous at all points when  $f$  is the function defined by

$$f(x) = \begin{cases} 3x-1 & \text{if } x \leq 2 \\ 9-2x & \text{if } x > 2 \end{cases}.$$

- In this case,  $f$  agrees with a polynomial on the open intervals  $(-\infty, 2)$  and  $(2, \infty)$ , so it is continuous on these intervals. To check continuity at  $y = 2$ , we note that

$$|f(x) - f(2)| = |f(x) - 5| = \begin{cases} 3|x-2| & \text{if } x \leq 2 \\ 2|x-2| & \text{if } x > 2 \end{cases}.$$

Given any  $\varepsilon > 0$ , we can then set  $\delta = \varepsilon/3$  to find that

$$|x-2| < \delta \implies |f(x) - f(2)| \leq 3|x-2| < 3\delta = \varepsilon.$$

3. Suppose that  $f$  is continuous with  $f(0) < 1$ . Show that there exists some  $\delta > 0$  such that  $f(x) < 1$  for all  $-\delta < x < \delta$ . Hint: use the  $\varepsilon$ - $\delta$  definition for some suitable  $\varepsilon$ .

- Since  $\varepsilon = 1 - f(0)$  is positive, some  $\delta > 0$  exists such that

$$|x-0| < \delta \implies |f(x) - f(0)| < \varepsilon \implies f(x) - f(0) < 1 - f(0).$$

In other words, one has  $f(x) < 1$  for all  $-\delta < x < \delta$ , as needed.

4. Show that  $4x^4 + 22x^2 \geq 4x^3 - 11$  for all  $x \in \mathbb{R}$ . Hint: you need to find a min.

- We need to show that  $f(x) = 4x^4 - 4x^3 + 22x^2 + 11$  is non-negative for all  $x$ . Let us then try to compute the minimum value of this function. We have

$$f'(x) = 16x^3 - 12x^2 + 44x = 4x(4x^2 - 3x + 11)$$

and the quadratic factor has no real roots because its discriminant is

$$\Delta = (-3)^2 - 4 \cdot 4 \cdot 11 < 0.$$

Thus, the quadratic factor is always positive, so  $f'(x) > 0$  if and only if  $x > 0$ . This shows that  $f$  is decreasing when  $x < 0$  and increasing when  $x > 0$ , so

$$\min f(x) = f(0) = 11.$$

5. Suppose that  $f$  is differentiable on  $[a, b]$  and that  $f'(x) > 0$  for all  $a \leq x \leq b$ . Show that if  $f(a) \leq y \leq f(b)$ , then there is exactly one  $x \in [a, b]$  such that  $f(x) = y$ .

- To prove the existence part, let  $g(x) = f(x) - y$  and note that

$$g(a) = f(a) - y \leq 0, \quad g(b) = f(b) - y \geq 0.$$

If either  $g(a) = 0$  or  $g(b) = 0$ , then we are done. Otherwise, the inequalities are both strict and we can apply Bolzano's theorem to find some  $c \in (a, b)$  such that  $g(c) = 0$ .

- To prove the uniqueness part, suppose  $f(x_1) = y = f(x_2)$  and  $x_1 \neq x_2$ . Then  $f'$  has a root by Rolle's theorem, contrary to the fact that  $f'$  is positive at all points.

6. Prove or give a counterexample to the following statements:

(a) If  $f$  is differentiable and strictly increasing on  $(a, b)$ , then  $f'(x) > 0$  for all  $a < x < b$ .

(b) If  $f$  is convex on  $(a, b)$ , then  $f''(x) \geq 0$  for all  $a < x < b$ .

- The first statement is false. For instance,  $f(x) = x^3$  is strictly increasing on  $(-1, 1)$  but its derivative  $f'(x) = 3x^2$  vanishes at  $x = 0$ .
- The second statement is also false. For instance, the absolute value function is convex but not differentiable at  $x = 0$ .

7. Prove that there is a function  $s$ , defined on all of  $\mathbb{R}$ , such that

$$s(0) = 0, \quad s'(x) = (1 + x^4)^{-1/2}.$$

Show that  $s$  is bounded.

- According to the fundamental theorem of calculus, the function

$$s(x) = \int_0^x \frac{dt}{\sqrt{1+t^4}}$$

is certainly such. Being differentiable, hence also continuous,  $s$  is bounded on  $[-1, 1]$ . To show that it is bounded when  $x > 1$ , we note that

$$\begin{aligned} |s(x)| &= \int_0^1 \frac{dt}{\sqrt{1+t^4}} + \int_1^x \frac{dt}{\sqrt{1+t^4}} \\ &\leq \int_0^1 dt + \int_1^x \frac{dt}{t^2} \\ &= 1 - \frac{1}{x} + 1 \end{aligned}$$

is strictly less than 2. Since  $s'(x)$  is even, it is easy to check that  $s(x)$  is odd. In fact, one can use the fundamental theorem of calculus to get

$$s(-x) = \int_0^{-x} s'(t) dt = \int_0^x s'(-u)(-du) = - \int_0^x s'(u) du = -s(x).$$

Thus,  $s(x)$  is odd and bounded when  $x > 1$ , so it is bounded when  $x < -1$  as well.

8. Prove that the following series diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^2}}, \quad \sum_{n=2}^{\infty} \frac{1}{n \log n}, \quad \sum_{n=1}^{\infty} \sin n.$$

- When it comes to the first series, we use the limit comparison test with

$$a_n = \frac{1}{\sqrt{1+n^2}}, \quad b_n = \frac{1}{n}.$$

Note that the limit comparison test does apply here because

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^2}{1+n^2}} = 1.$$

Since  $\sum_{n=1}^{\infty} b_n$  is a divergent  $p$ -series, the series  $\sum_{n=1}^{\infty} a_n$  must also diverge.

- To test the second series for convergence, we use the integral test. Note that

$$f(n) = \frac{1}{n \log n}$$

is clearly decreasing with  $f(n) \rightarrow 0$  as  $n \rightarrow \infty$ . In addition, we have

$$\int_e^n f(x) dx = \int_e^n \frac{dx}{x \log x} = \int_1^{\log n} \frac{du}{u} = \log \log n,$$

so this integral is not bounded for all  $n$  and the second series diverges.

- For the last series, we show that the  $n$ th term fails to approach zero. Suppose

$$\lim_{n \rightarrow \infty} \sin n = 0 \tag{1}$$

for the sake of contradiction. Using the addition formula, we then get

$$\lim_{n \rightarrow \infty} \sin 1 \cdot \cos n = \lim_{n \rightarrow \infty} \sin(n + 1) - \sin n \cdot \cos 1 = 0.$$

Since  $\sin x$  is positive on  $(0, \pi)$ , we have  $\sin 1 > 0$  and so

$$\lim_{n \rightarrow \infty} \cos n = 0 \tag{2}$$

by above. Combining equations (1) and (2), however, gives the contradiction

$$0 = \lim_{n \rightarrow \infty} \sin^2 n + \cos^2 n = 1.$$

Thus, equation (1) cannot hold and the last series diverges by the  $n$ th term test.