MA121, 2009 Final exam
Solutions

1. *Show that there exists some* $0 < x < 1$ *such that* $4x^3 + 3x = 2x^2 + 2$.

- Let $f(x) = 4x^3 + 3x - 2x^2 - 2$ for all $x \in [0, 1]$. Being a polynomial, f is continuous on the closed interval $[0, 1]$ and we also have
$$f(0) = -2 < 0, \qquad f(1) = 4 + 3 - 2 - 2 = 3 > 0.$$
In view of Bolzano's theorem, this means that $f(x) = 0$ for some $x \in (0, 1)$.

2. *Suppose* f *is a differentiable function with* $|f'(x)| \leq 1$ *for all* $x \in \mathbb{R}$. *Show that*
$$|f(x) - f(y)| \leq |x - y| \qquad \text{for all } x, y \in \mathbb{R}.$$

- When $x = y$, the desired inequality states that $0 \leq 0$, so it is certainly true.
- Suppose now that $x \neq y$. According to the mean value theorem, we must then have
$$\frac{f(x) - f(y)}{x - y} = f'(c) \quad \Longrightarrow \quad |f(x) - f(y)| = |x - y| \cdot |f'(c)|$$
for some c between x and y. Using the fact that $|f'(c)| \leq 1$, we conclude that
$$|f(x) - f(y)| = |x - y| \cdot |f'(c)| \leq |x - y|.$$

3. *Suppose* f *is a differentiable function with* $f'(x) = f(x) + e^x$ *for all* $x \in \mathbb{R}$. *Show that there exists some constant* C *such that* $f(x) = xe^x + Ce^x$ *for all* $x \in \mathbb{R}$.

- Letting $g(x) = f(x)e^{-x} - x$ for convenience, one easily finds that
$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} - 1 = f(x)e^{-x} + 1 - f(x)e^{-x} - 1 = 0.$$
In particular, $g(x)$ is actually constant, say $g(x) = C$ for all $x \in \mathbb{R}$, and thus
$$g(x) = C \quad \Longrightarrow \quad f(x)e^{-x} = x + C \quad \Longrightarrow \quad f(x) = xe^x + Ce^x.$$

4. *Compute each of the following integrals:*
$$\int \frac{3x + 2}{(x + 1)^2}\, dx, \qquad \int \log(x^2 - 1)\, dx.$$

- To compute the first integral, we use the substitution $u = x + 1$ to get
$$\begin{aligned} \int \frac{3x + 2}{(x + 1)^2}\, dx = \int \frac{3u - 1}{u^2}\, du &= \int 3u^{-1} - u^{-2}\, du \\ &= 3 \log|u| + u^{-1} + C \\ &= 3 \log|x + 1| + (x + 1)^{-1} + C. \end{aligned}$$

- To compute the second integral, we integrate by parts to find that

$$\int \log(x^2-1)\,dx = x\log(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx.$$

Using division of polynomials to simplify the rational function, we get

$$\frac{2x^2}{x^2-1} = 2 + \frac{2}{x^2-1} \quad\Longrightarrow\quad \int \frac{2x^2}{x^2-1}\,dx = 2x + \int \frac{2}{x^2-1}\,dx$$

and we can now use partial fractions to write

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad\Longrightarrow\quad 2 = A(x+1) + B(x-1).$$

Setting $x=1$ and $x=-1$ gives $2=2A$ and $2=-2B$, respectively, hence

$$\begin{aligned}\int \frac{2x^2}{x^2-1}\,dx &= 2x + \int \frac{1}{x-1} - \frac{1}{x+1}\,dx \\ &= 2x + \log|x-1| - \log|x+1| + C.\end{aligned}$$

In particular, the original integral is given by

$$\int \log(x^2-1)\,dx = x\log(x^2-1) - 2x - \log|x-1| + \log|x+1| + C.$$

5. *Show that* $\log x \leq x-1$ *for all* $x>0$.

- Letting $f(x) = \log x - x + 1$ for convenience, one easily finds that

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}\,.$$

Thus, $f'(x)$ is positive if and only if $1-x>0$, hence if and only if $x<1$. This shows that f is increasing when $x<1$ and also decreasing when $x>1$, so

$$\max f(x) = f(1) = \log 1 - 1 + 1 = 0 \quad\Longrightarrow\quad f(x) \leq \max f(x) = 0.$$

6. *Test each of the following series for convergence:*

$$\sum_{n=1}^\infty \frac{e^{1/n}}{n}\,, \qquad \sum_{n=1}^\infty \frac{1}{n^n}\,, \qquad \sum_{n=1}^\infty \frac{(-1)^n\,\sin n}{n^2}\,.$$

- To test the first series for convergence, we use the comparison test to get

$$\sum_{n=1}^\infty \frac{e^{1/n}}{n} > \sum_{n=1}^\infty \frac{1}{n}\,.$$

Being larger than a divergent p-series, the given series must thus diverge.

- To test the second series for convergence, one can use the ratio test to get

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0;$$

this implies convergence due to the ratio test. Alternatively, one can argue that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

and that this implies convergence due to the comparison test.

- Finally, the last series converges absolutely because we have

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the last series converges absolutely, it must thus converge as well.

7. *Find the maximum value of $f(x,y) = xy^2$ over the closed unit disk*

$$D = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\}.$$

- Since $f_x = y^2$ and $f_y = 2xy$, every point on the x-axis is critical and we have

$$f(x,0) = 0.$$

Next, we check the points on the boundary of the disk. Along the boundary,

$$y^2 = 1 - x^2 \quad \Longrightarrow \quad f(x,y) = xy^2 = x - x^3$$

and we need to find the maximum value of this function on $[-1,1]$. Noting that

$$g(x) = x - x^3 \quad \Longrightarrow \quad g'(x) = 1 - 3x^2,$$

we see that the maximum value may only occur at $x = \pm 1$ or $x = \pm 1/\sqrt{3}$. Since

$$g(\pm 1) = 0, \qquad g(\pm 1/\sqrt{3}\,) = \pm\frac{2\sqrt{3}}{9},$$

the largest value we have found so far is the value $2\sqrt{3}/9$ obtained above.

8. *Compute the double integral*

$$\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\, dx\, dy.$$

- In this case, switching the order of integration gives

$$\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\,dx\,dy = \int_0^2 \int_0^{x^2} \cos(x^3)\,dy\,dx = \int_0^2 x^2 \cos(x^3)\,dx.$$

  Using the substitution $u = x^3$, we now get $du = 3x^2\,dx$ and this implies that

$$\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\,dx\,dy = \frac{1}{3}\int_0^8 \cos u\,du = \left[\frac{\sin u}{3}\right]_0^8 = \frac{\sin 8}{3}.$$