

**MA121, 2008 Final exam**  
**Solutions**

1. Suppose that  $A$  is a nonempty subset of  $\mathbb{R}$  that has a lower bound and let  $\varepsilon > 0$  be given. Show that there exists an element  $a \in A$  such that  $\inf A \leq a < \inf A + \varepsilon$ .

- Note that  $\inf A + \varepsilon$  cannot be a lower bound of  $A$  because it is larger than the greatest lower bound of  $A$ . This means that some  $a \in A$  is such that  $a < \inf A + \varepsilon$ . On the other hand, we must also have  $a \geq \inf A$  because  $a \in A$  and  $\inf A$  is a lower bound of  $A$ .

2. Show that the polynomial  $f(x) = x^3 - 7x^2 - 5x + 1$  has exactly one root in  $[0, 2]$ .

- Being a polynomial,  $f$  is continuous on the closed interval  $[0, 2]$  and we also have

$$f(0) = 1 > 0, \quad f(2) = 8 - 28 - 10 + 1 = -29 < 0.$$

Thus,  $f$  has a root in  $[0, 2]$  by Bolzano's theorem. Suppose it has two roots in  $[0, 2]$ . In view of Rolle's theorem,  $f'$  must then have a root in  $[0, 2]$  as well. On the other hand,

$$f'(x) = 3x^2 - 14x - 5$$

and the roots of this function are given by the quadratic formula

$$x = \frac{14 \pm \sqrt{14^2 + 4 \cdot 3 \cdot 5}}{2 \cdot 3} = \frac{14 \pm 16}{6} \implies x = 5, \quad x = -1/3.$$

Since none of those lies in  $[0, 2]$ , we conclude that  $f$  cannot have two roots in  $[0, 2]$ .

3. Find the maximum value of  $f(x) = \frac{x+1}{x^2+8}$  over the closed interval  $[0, 3]$ .

- Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which  $f'$  does not exist and the points at which  $f'$  is equal to zero. In this case,

$$f'(x) = \frac{x^2 + 8 - 2x \cdot (x + 1)}{(x^2 + 8)^2} = -\frac{x^2 + 2x - 8}{(x^2 + 8)^2} = -\frac{(x + 4)(x - 2)}{(x^2 + 8)^2}$$

and so the only points at which the maximum value may occur are

$$x = -4, \quad x = 2, \quad x = 0, \quad x = 3.$$

We exclude the leftmost point, which fails to lie in  $[0, 3]$ , and we now compute

$$f(2) = \frac{3}{12} = \frac{1}{4}, \quad f(0) = \frac{1}{8}, \quad f(3) = \frac{4}{17}.$$

Based on these observations, we deduce that the maximum value is  $f(2) = 1/4$ .

4. Compute each of the following integrals:

$$\int \frac{6x + 9}{x^3 + 3x^2} dx, \quad \int 2x^3 e^{x^2} dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{6x + 9}{x^3 + 3x^2} = \frac{6x + 9}{x^2(x + 3)} = \frac{Ax + B}{x^2} + \frac{C}{x + 3} \quad (*)$$

for some constants  $A, B, C$  that need to be determined. Clearing denominators gives

$$6x + 9 = (Ax + B)(x + 3) + Cx^2$$

and we can now look at some suitable choices of  $x$  to find

$$\begin{array}{llll} x = -3 & \implies & -9 = 9C & \implies & C = -1, \\ x = 0 & \implies & 9 = 3B & \implies & B = 3, \\ x = -1 & \implies & 3 = -2A + 2B + C & \implies & A = 1. \end{array}$$

Returning to equation  $(*)$ , we now get

$$\frac{6x + 9}{x^3 + 3x^2} = \frac{x + 3}{x^2} - \frac{1}{x + 3} = \frac{1}{x} + \frac{3}{x^2} - \frac{1}{x + 3}$$

and we may integrate this equation term by term to conclude that

$$\int \frac{6x + 9}{x^3 + 3x^2} dx = \log |x| - 3x^{-1} - \log |x + 3| + C.$$

- For the second integral, we use the substitution  $u = x^2$ . This gives  $du = 2x dx$ , hence

$$\int 2x^3 e^{x^2} dx = \int 2x \cdot x^2 e^{x^2} dx = \int u e^u du.$$

Focusing on the rightmost integral, we integrate by parts to find that

$$\int u e^u du = \int u (e^u)' du = u e^u - \int e^u du = u e^u - e^u + C.$$

Once we now combine the last two equations, we get

$$\int 2x^3 e^{x^2} dx = \int u e^u du = u e^u - e^u + C = x^2 e^{x^2} - e^{x^2} + C.$$

5. Suppose  $f$  is continuous on  $[a, b]$ . Show that there exists some  $c \in (a, b)$  such that

$$\int_a^b f(t) dt = (b - a) \cdot f(c).$$

As a hint, apply the mean value theorem to the function  $F(x) = \int_a^x f(t) dt$ .

- According to the mean value theorem, there exists some  $c \in (a, b)$  such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

In addition, we have  $F'(x) = f(x)$  for all  $x$ , and we also have

$$F(a) = \int_a^a f(t) dt = 0, \quad F(b) = \int_a^b f(t) dt.$$

Once we now combine all these facts, we may conclude that

$$F(b) - F(a) = (b - a) \cdot F'(c) \implies \int_a^b f(t) dt = (b - a) \cdot f(c).$$

6. Test each of the following series for convergence:

$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^3 + n}, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

- To test the first series for convergence, we use the limit comparison test with

$$a_n = \frac{n^2 + 2}{n^3 + n}, \quad b_n = \frac{n^2}{n^3} = \frac{1}{n}.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 + 2}{n^3 + n} \cdot n = \lim_{n \rightarrow \infty} \frac{n^2 + 2}{n^2 + 1} = 1.$$

Since the series  $\sum_{n=1}^{\infty} b_n$  is a divergent  $p$ -series, the series  $\sum_{n=1}^{\infty} a_n$  must also diverge.

- To test the second series for convergence, we use the ratio test. In this case, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left( \frac{n}{n+1} \right)^n$$

and this implies that

$$L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e}.$$

Since  $e > 1$ , this limit is strictly less than 1 and so the given series converges.

7. Let  $f$  be the function defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Show that  $f$  is not integrable on any closed interval  $[a, b]$ .

- Given a partition  $P = \{x_0, x_1, \dots, x_n\}$  of the interval  $[a, b]$ , one easily finds that

$$\begin{aligned} S^-(f, P) &= \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= 0(x_1 - x_0) + 0(x_2 - x_1) + 0(x_3 - x_2) + \dots + 0(x_n - x_{n-1}) = 0, \end{aligned}$$

hence  $\sup S^-(f, P) = 0$  as well. On the other hand, one also has

$$\begin{aligned} S^+(f, P) &= \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= 1(x_1 - x_0) + 1(x_2 - x_1) + 1(x_3 - x_2) + \dots + 1(x_n - x_{n-1}) \\ &= x_n - x_0 = b - a \end{aligned}$$

so that  $\inf S^+(f, P) = b - a$  as well. This gives  $\sup S^- \neq \inf S^+$  because  $b - a \neq 0$ .

8. Suppose that  $z = z(r, s, t)$ , where  $r = u - v$ ,  $s = v - w$  and  $t = w - u$ . Assuming that all partial derivatives exist, show that  $z_u + z_v + z_w = 0$ .
- Using the definitions of  $r, s, t$  together with the chain rule, we get

$$\begin{aligned} z_u &= z_r r_u + z_s s_u + z_t t_u = z_r - z_t \\ z_v &= z_r r_v + z_s s_v + z_t t_v = -z_r + z_s \\ z_w &= z_r r_w + z_s s_w + z_t t_w = -z_s + z_t. \end{aligned}$$

Adding these three equations, one now finds that  $z_u + z_v + z_w = 0$ , indeed.

9. Classify the critical points of the function defined by  $f(x, y) = 3xy - x^3 - y^3$ .

- To find the critical points, we need to solve the equations

$$\begin{aligned} 0 &= f_x(x, y) = 3y - 3x^2 = 3(y - x^2), \\ 0 &= f_y(x, y) = 3x - 3y^2 = 3(x - y^2). \end{aligned}$$

These give  $y = x^2$  and also  $x = y^2$ , so we easily get

$$x = y^2 = x^4 \implies x^4 - x = 0 \implies x(x^3 - 1) = 0 \implies x = 0, 1.$$

In particular, the only critical points are  $(0, 0)$  and  $(1, 1)$ .

- In order to classify the critical points, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}.$$

When it comes to the critical point  $(0, 0)$ , this gives

$$H = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \implies \det H = -9 < 0$$

so the origin is a saddle point. When it comes to the critical point  $(1, 1)$ , we have

$$H = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \implies \det H = 36 - 9 > 0$$

and also  $f_{xx} = -6 < 0$ , so this critical point is a local maximum.

10. Compute the double integral

$$\int_0^1 \int_y^1 e^{x^2} dx dy.$$

- To compute the given integral, we switch the order of integration to get

$$\int_0^1 \int_y^1 e^{x^2} dx dy = \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 x e^{x^2} dx.$$

Using the substitution  $u = x^2$ , we now get  $du = 2x dx$ , and this implies that

$$\int_0^1 \int_y^1 e^{x^2} dx dy = \frac{1}{2} \int_0^1 e^u du = \left[ \frac{e^u}{2} \right]_0^1 = \frac{e - 1}{2}.$$