

**MA121, 2007 Final exam**  
**Solutions**

1. Suppose that  $A$  is a nonempty subset of  $\mathbb{R}$  that has an upper bound, and let  $B$  be the set of all upper bounds of  $A$ . Show that  $\inf B = \sup A$ .

- Since  $\sup A$  is the least upper bound of  $A$ , it is also the least element of  $B$ , namely

$$\sup A = \min B.$$

Since  $B$  has a minimum, however, it also has an infimum and the two are equal, so

$$\inf B = \min B = \sup A.$$

2. Let  $a \in \mathbb{R}$  be a given number and let  $f$  be the function defined by

$$f(x) = \left\{ \begin{array}{ll} ax^2 + 2x & \text{if } x \neq 2 \\ 2a + 8 & \text{if } x = 2 \end{array} \right\}.$$

Find the value of  $a$  for which  $f$  is continuous at  $y = 2$ .

- To say that  $f$  is continuous at  $y = 2$  is to say that

$$\lim_{x \rightarrow 2} f(x) = f(2).$$

In our case, the left hand side is equal to

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (ax^2 + 2x) = a \cdot 2^2 + 2 \cdot 2 = 4a + 4,$$

while the right hand side is equal to  $f(2) = 2a + 8$ . In particular, we have

$$\lim_{x \rightarrow 2} f(x) = f(2) \iff 4a + 4 = 2a + 8 \iff a = 2.$$

3. Find the minimum value of  $f(x) = x^4 + 4x^3 - 8x^2 + 2$  over the whole real line.

- The derivative of the given function is

$$f'(x) = 4x^3 + 12x^2 - 16x = 4x(x^2 + 3x - 4) = 4x(x - 1)(x + 4)$$

and we can determine the sign of  $f'$  using the table below.

$x$	$-4$	$0$	$1$	
$4x$	$-$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
$x + 4$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

According to the table, the minimum value of  $f$  can now be found by comparing

$$f(-4) = 4^4 - 4 \cdot 4^3 - 8 \cdot 4^2 + 2 = -126, \quad f(1) = 1 + 4 - 8 + 2 = -1.$$

Since the former is smaller and also attained, this means that  $\min f(x) = -126$ .

4. Let  $a, b, c \in \mathbb{R}$  be some fixed constants such that  $\frac{a}{3} + \frac{b}{2} + c = 0$ . Show that

$$ax^2 + bx + c = 0 \quad \text{for some } x \in (0, 1).$$

As a hint, apply the mean value theorem to a function whose derivative is  $ax^2 + bx + c$ .

- Following the hint, let us consider the function

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx.$$

Then  $f$  is differentiable on  $[0, 1]$  with  $f'(x) = ax^2 + bx + c$  for all  $x$ , and we also have

$$f(0) = 0, \quad f(1) = \frac{a}{3} + \frac{b}{2} + c = 0.$$

Using the mean value theorem, we conclude that some  $x \in (0, 1)$  exists such that

$$f'(x) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 0}{1 - 0} = 0 \quad \implies \quad ax^2 + bx + c = 0.$$

5. Suppose that  $f$  is a function which satisfies the inequality

$$|f(x) - f(y)| \leq |x - y|^2 \quad \text{for all } x, y \in \mathbb{R}.$$

Show that  $f$  is actually constant.

- We need only show that  $f'(y) = 0$  for all  $y \in \mathbb{R}$ . Using the given inequality, we get

$$0 \leq \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y| \quad \text{whenever } x \neq y.$$

Since  $|x - y|$  approaches zero as  $x \rightarrow y$ , the quotient above is thus squeezed between two functions which approach zero as  $x \rightarrow y$ . In view of the Squeeze Law, the quotient itself must approach zero as  $x \rightarrow y$ . This also implies that  $f'(y) = 0$ , as needed.

6. Evaluate each of the following integrals:

$$\int \frac{4x^2 - 15x + 12}{x^3 - 5x^2 + 6x} dx, \quad \int \frac{x^3 - x + 1}{x + 1} dx.$$

As a hint for the first integral, you might want to factor the denominator.

- To evaluate the first integral, we use partial fractions to write

$$\frac{4x^2 - 15x + 12}{x^3 - 5x^2 + 6x} = \frac{4x^2 - 15x + 12}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

for some constants  $A, B, C$  that need to be determined. Clearing denominators, we get

$$4x^2 - 15x + 12 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2)$$

and we can now look at some suitable choices of  $x$  to find that

$$x = 0, 2, 3 \implies 12 = 6A, \quad -2 = -2B, \quad 3 = 3C.$$

This gives  $A = 2$  and  $B = C = 1$ , so the partial fractions decomposition reads

$$\frac{4x^2 - 15x + 12}{x^3 - 5x^2 + 6x} = \frac{2}{x} + \frac{1}{x-2} + \frac{1}{x-3}.$$

Once we now integrate this equation term by term, we get

$$\int \frac{4x^2 - 15x + 12}{x^3 - 5x^2 + 6x} dx = 2 \log |x| + \log |x-2| + \log |x-3| + C.$$

- For the second integral, we use division of polynomials to write

$$\frac{x^3 - x + 1}{x+1} = x^2 - x + \frac{1}{x+1}.$$

Integrating this equation term by term, we then easily find that

$$\int \frac{x^3 - x + 1}{x+1} dx = \frac{x^3}{3} - \frac{x^2}{2} + \log |x+1| + C.$$

7. Test each of the following series for convergence:

$$\sum_{n=0}^{\infty} \frac{n!}{(2n)!}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \quad \sum_{n=1}^{\infty} \frac{n^2 + 2}{n^3 + n}.$$

- To test the first series for convergence, we use the ratio test. Since the limit

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} = \lim_{n \rightarrow \infty} \frac{n+1}{(2n+1)(2n+2)} = 0$$

is strictly less than 1, the first series converges by the ratio test.

- For the second series, we use the alternating series test with

$$a_n = \frac{1}{n} = n^{-1}.$$

Note that  $a_n$  is non-negative for each  $n \geq 1$  and that  $a_n$  is decreasing because

$$a'_n = -n^{-2} < 0.$$

Since  $a_n = 1/n$  approaches zero as  $n \rightarrow \infty$ , we see that the second series converges.

- For the last series, we use the limit comparison test with

$$a_n = \frac{n^2 + 2}{n^3 + n}, \quad b_n = \frac{n^2}{n^3} = \frac{1}{n}.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 + 2}{n^3 + n} \cdot \frac{n}{1} = \lim_{n \rightarrow \infty} \frac{n^2 + 2}{n^2 + 1} = 1.$$

Since the series  $\sum_{n=1}^{\infty} b_n$  is a divergent  $p$ -series, the series  $\sum_{n=1}^{\infty} a_n$  diverges as well.

8. Evaluate each of the following sums:

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+2}}, \quad \sum_{n=2}^{\infty} \frac{e^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n 9^{n+1}}{(2n)!}.$$

- The first sum is related to a geometric series, namely

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+2}} = \frac{2}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{9} \cdot \frac{1}{1 - 2/3} = \frac{6}{9} = \frac{2}{3}.$$

- Relating the second sum to the Taylor series for the exponential function, we get

$$\sum_{n=2}^{\infty} \frac{e^n}{n!} = \sum_{n=0}^{\infty} \frac{e^n}{n!} - 1 - e = e^e - 1 - e.$$

- Finally, the third sum is related to the Taylor series for the cosine function, namely

$$\sum_{n=1}^{\infty} \frac{(-1)^n 9^{n+1}}{(2n)!} = 9 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} = 9(\cos 3 - 1).$$

9. Let  $f$  be the function defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}.$$

Show that  $f$  is integrable on  $[0, 1]$ .

- Since  $f(x) = 1$  at all points except for  $x = 0$ , it should be clear that

$$\begin{aligned} S^+(f, P) &= \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1}) = x_n - x_0 = 1 \end{aligned}$$

for all partitions  $P = \{x_0, x_1, \dots, x_n\}$  of the closed interval  $[0, 1]$ .

- Since  $[x_0, x_1]$  is the only subinterval that contains the point  $x = 0$ , we also have

$$\begin{aligned} S^-(f, P) &= \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= (x_2 - x_1) + (x_3 - x_2) + \dots + (x_n - x_{n-1}) = x_n - x_1 = 1 - x_1. \end{aligned}$$

Taking the supremum over all possible partitions, we may thus conclude that

$$\sup_P \{S^-(f, P)\} = \sup_{0 < x_1 < 1} (1 - x_1) = 1 = \inf_P \{S^+(f, P)\}.$$

10. Define a sequence  $\{a_n\}$  by setting  $a_1 = 2$  and

$$a_{n+1} = \frac{1}{3 - a_n} \quad \text{for each } n \geq 1.$$

Show that  $0 < a_{n+1} \leq a_n \leq 2$  for each  $n \geq 1$ . Use this fact to conclude that the sequence converges and then find its limit.

- Since the first two terms are  $a_1 = 2$  and  $a_2 = 1$ , the statement

$$0 < a_{n+1} \leq a_n \leq 2$$

does hold when  $n = 1$ . Suppose that it holds for some  $n$ , in which case

$$\begin{aligned} 0 > -a_{n+1} \geq -a_n \geq -2 &\implies 3 > 3 - a_{n+1} \geq 3 - a_n \geq 1 \\ &\implies 1/3 < a_{n+2} \leq a_{n+1} \leq 1. \end{aligned}$$

Thus, the statement holds for  $n + 1$  as well, so it must actually hold for all  $n \in \mathbb{N}$ . This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by  $L$ . Using the definition of the sequence, we then find that

$$a_{n+1} = \frac{1}{3 - a_n} \implies L = \frac{1}{3 - L} \implies L^2 - 3L + 1 = 0.$$

Solving this quadratic equation now gives

$$L = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Since  $0 < a_n \leq 2$  for each  $n \in \mathbb{N}$ , however, we must also have  $0 \leq L \leq 2$ , so

$$L = \frac{3 - \sqrt{5}}{2}.$$