ASSIGNMENT 3 - SOLUTIONS

1. Determine the nature of the critical points of
   
   \( (a) \ f(x, y) = e^{xy} ; \)
   \( (b) \ f(x, y) = x^2 + 2y^2 - xy ; \)
   \( (c) \ f(x, y) = 3xe^y - x^3 - e^{3y}. \)

Solution. \( (a) \) We have \( (f_x, f_y) = (ye^{xy}, xe^{xy}) \). The critical points are the points where \( (f_x, f_y) = (0, 0) \), that is,

\[
\begin{align*}
ye^{xy} &= 0 \\
x e^{xy} &= 0
\end{align*}
\]
so that \( (x, y) = (0, 0) \).

The second order partial derivatives are

\[
\begin{align*}
f_{xx}(x, y) &= y^2 e^{xy} , \\
f_{xy}(x, y) &= e^{xy} + xy e^{xy} , \\
f_{yy}(x, y) &= x^2 e^{xy} .
\end{align*}
\]

Then

\[
f_{xx}(0, 0) = f_{yy}(0, 0) = 0 , \\
f_{xy}(0, 0) = 1 ,
\]
so that

\[
D(0, 0) = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = -1 < 0
\]
which implies that \( (0, 0) \) is a saddle point.

\( (b) \) We have \( (f_x, f_y) = (2x - y, 4y - x) \). The critical points are the points where \( (f_x, f_y) = (0, 0) \), that is,

\[
\begin{align*}
2x - y &= 0 \\
4y - x &= 0
\end{align*}
\]
or \( (x, y) = (0, 0) \).

The second order partial derivatives are

\[
\begin{align*}
f_{xx}(0, 0) &= 2 , \\
f_{xy}(0, 0) &= -1 , \\
f_{yy}(0, 0) &= 4 ,
\end{align*}
\]
and

\[
D(0, 0) = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 7 > 0 .
\]

Since \( f_{xx}(0, 0) = 2 > 0 \) and \( D(0, 0) > 0 \) we deduce that \( (0, 0) \) is a local minimum point\(^1\) for \( f \).

\( (c) \) We have \( (f_x, f_y) = (3e^y - 3x^2, 3xe^y - 3e^{3y}) \). The critical points are obtained by solving the system

\[
\begin{align*}
3e^y - 3x^2 &= 0 \\
3xe^y - 3e^{3y} &= 0
\end{align*}
\]

Divide the second equation by \( 3e^y \) to obtain \( x = e^{2y} > 0 \). Substituting \( x = e^{2y} \) in the first equation we get \( e^y = e^{4y} \), which implies \( y = 0 \). Then \( x = e^{2y} = 1 \). Thus \( (1, 0) \) is the only critical point.

The second order partial derivatives are

\[
\begin{align*}
f_{xx}(x, y) &= -6x , \\
f_{xy}(x, y) &= 3e^y , \\
f_{yy}(x, y) &= 3xe^y - 9e^{3y} .
\end{align*}
\]

\(^1\)It is actually an absolute minimum since

\[
x^2 - xy + 2y^2 = (x - \frac{1}{2} y)^2 + \frac{7}{4} y^2 \geq 0 = f(0, 0) .
\]
Then

\[ D(1, 0) = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 > 0. \]

Since \( f_{xx}(1, 0) = -6 < 0 \) and \( D(1, 0) > 0 \) we deduce that \((1, 0)\) is a local maximum point for \( f \).

2. Let \( f(x, y) = x^2 + 4y^2 - 4xy \). Can you say whether the critical points are local minima, maxima or saddle points?

Solution. We have \((f_x, f_y) = (2x - 4y, 8y - 4x)\). The critical points are obtained by solving the system

\[
\begin{align*}
2x - 4y &= 0 \\
8y - 4x &= 0
\end{align*}
\]

so \( x = 2y \).

The critical points are therefore \((x, \frac{x}{2})\) with \( x \in \mathbb{R} \). The second order partial derivatives test is inconclusive since

\[ D(x, \frac{x}{2}) = f_{xx}(x, x/2) f_{yy}(x, x/2) - [f_{xy}(x, x/2)]^2 = 2 \cdot 8 - (-4)^2 = 0. \]

However,

\[ f(x, y) = (x - 2y)^2 \geq 0 = f(x, \frac{x}{2}) \]

shows that all points \((x, \frac{x}{2})\) with \( x \in \mathbb{R} \) are absolute minimum points.

3. Find the points on the hyperboloid

\[ z^2 - \frac{y^2}{2} - \frac{x^2}{3} = 1 \]

that are closest to the origin.

Solution. The distance from a point \((x, y, z)\) to \((0, 0, 0)\) is \( \sqrt{x^2 + y^2 + z^2} \). Let us therefore minimize the function \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraint \( g(x, y, z) = 1 \), where

\[ g(x, y, z) = z^2 - \frac{y^2}{2} - \frac{x^2}{3}. \]

The minimum points are found among the points on the hyperboloid where

\[ \nabla f(x, y, z) = \lambda \nabla g(x, y, z), \]

that is,

\[
\begin{align*}
(2x, 2y, 2z) &= \lambda \left( \frac{-2x}{3}, -y, 2z \right), \\
z^2 - \frac{y^2}{2} - \frac{x^2}{3} &= 1.
\end{align*}
\]

or

\[
\begin{align*}
x(1 + \frac{4}{3}) &= 0, \\
y(\lambda + 2) &= 0, \\
z(\lambda - 1) &= 0, \\
z^2 - \frac{y^2}{2} - \frac{x^2}{3} &= 1.
\end{align*}
\]

From the first equation we deduce that either \( x = 0 \) or \( \lambda = -3 \).

If \( \lambda = -3 \) the system becomes

\[
\begin{align*}
0 &= 0, \\
-y &= 0, \\
-4z &= 0, \\
z^2 - \frac{y^2}{2} - \frac{x^2}{3} &= 1.
\end{align*}
\]
Hence \( y = z = 0 \) and the last equation above becomes \(-\frac{x^2}{2} = 1\), which is a contradiction. Thus there are no solutions for \( \lambda = -3 \).

If \( x = 0 \) the system becomes

\[
\begin{align*}
    x &= 0 \\
 y(\lambda + 2) &= 0 \\
 z(\lambda - 1) &= 0 \\
 z^2 - \frac{y^2}{2} &= 1.
\end{align*}
\]

In this case either \( y = 0 \) or \( \lambda = -2 \). For \( \lambda = -2 \) we obtain \( z = 0 \) and then the last equation above becomes \(-\frac{y^2}{2} = 1\), which is again a contradiction. On the other hand, if \( y = 0 \) we get \( z^2 = 1 \), which implies \( z = \pm 1 \). Then \((x, y, z) = (0, 0, \pm 1)\). Both these points have the distance to the origin equal to 1. Thus the points \((x, y, z) = (0, 0, \pm 1)\) are the closest to the origin.

4. Find the minimum value of \( x^2 + y^2 + z^2 \) subject to the constraints \( x + y - z = 0 \) and \( x + 3y + z = 2 \). Give a geometrical interpretation of your answer and using it explain why \( x^2 + y^2 + z^2 \) has no maximum subject to these constraints.

Solution. Let

\[
f(x, y, z) = x^2 + y^2 + z^2, \quad g(x, y, z) = x + y - z, \quad h(x, y, z) = x + 3y + z.
\]

The minimum points are found among the points with

\[
\nabla f = \lambda \nabla g + \mu \nabla h,
\]

that is,

\[
\begin{align*}
    (2x, 2y, 2z) &= \lambda (1, 1, -1) + \mu (1, 3, 1), \\
    x + y - z &= 0, \\
    x + 3y + z &= 2.
\end{align*}
\]

Then

\[
\begin{align*}
    x &= \frac{\lambda + \mu}{2} \\
    y &= \frac{\lambda + 3\mu}{2} \\
    z &= \frac{-\lambda + \mu}{2} \\
    x + y - z &= 0, \\
    x + 3y + z &= 2.
\end{align*}
\]

Substituting the values of \( x, y, z \) in the last two equalities above, we get

\[
x + y - z = \frac{3\lambda + 3\mu}{2} = 0 \quad \text{and} \quad x + 3y + z = \frac{3\lambda + 11\mu}{2} = 2.
\]

Hence \( \lambda = -\mu \) and \( 3\lambda + 11\mu = 4 \). Thus \( \lambda = -\frac{1}{2} \) and \( \mu = \frac{1}{2} \). Returning to (*) we obtain \( x = 0 \) and \( y = z = \frac{1}{2} \). We compute \( f(0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \).

Let us now give a geometrical interpretation of our problem. We want to find the extrema of \( x^2 + y^2 + z^2 \) (that is, the square of the distance from \((x, y, z)\) to the origin) for \((x, y, z)\) lying in the intersection of the two planes \( x + y - z = 0 \) and \( x + 3y + z = 2 \), which is a line that we shall call \( L \). There is obviously no maximum distance from the origin to the given line but there is certainly a minimum distance. The square of the minimum distance is \( \frac{1}{2} \) and is obtained for \((x, y, z) = (0, \frac{1}{2}, \frac{1}{2})\).