CLASSICAL FIELD THEORY AND ELECTRODYNAMICS

Summary of Formulæ and their Applications

1. Particle Dynamics. The Lorentz scalar action describing a particle of electric charge q and four momentum p_μ ($\mu \in \{0, 1, 2, 3\}$) interacting with an electromagnetic four potential A_μ

$$S \;=\; -\int \left(p_\mu + \frac{q}{c} A_\mu \right) dx^\mu$$

gives rise, through Hamilton's variational principle, to the following particle equation of motion

$$\frac{dp_\mu}{d\tau} \;=\; \frac{q}{c}\, F_{\mu\nu}\, \frac{dx^\nu}{d\tau}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with ∂_μ referring to the partial derivative $\partial/\partial x^\mu$ and where the relativistically invariant time increment $d\tau$ is given by $c^2 d\tau^2 \;=\; c^2 dt^2 \;-\; d\vec{x}^{\,2} \;=\; dx_\mu\, dx^\mu$.

2. Vector Field Dynamics. In the Euler-Lagrange equation of motion for a scalar field $\varphi(x^\rho)$

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} \right] \;-\; \frac{\partial \mathcal{L}}{\partial \varphi} \;=\; 0$$

the variable φ may be replaced by the vector potential A_ν to formulate an equation of motion for a vector field. The Euler-Lagrange equation for the invariant Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \;-\; \tfrac{1}{c} J^\sigma A_\sigma \;=\; \tfrac{1}{4} F_\rho^{\;\sigma} F_\sigma^{\;\rho} \;-\; \tfrac{1}{c} A_\sigma J^\sigma$$

leads to the inhomogeneous and homogeneous Maxwell equations — and charge conservation

$$\partial_\mu F^{\mu\nu} \;=\; c^{-1} J^\nu \,, \qquad \partial_\mu \widetilde{F}^{\mu\nu} \;=\; 0^\nu \,, \qquad \partial_\nu J^\nu \;=\; 0 \,,$$

where the 4-current density J^ν has c times the charge density ρ as its zero component and the three-current density $\vec{J}$ as its 3-vector component. Heaviside-Lorentz units are used throughout.

3. Dual Tensor Components. Particular values $\varepsilon^{123} = \epsilon^{0123} = 1$ and $\epsilon_{0123} = -1$ normalise the antisymmetric Levi-Civita symbols. Components of the electromagnetic field tensor $F^{\mu\nu}$ and its dual tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ may be related to elements of the electric and magnetic fields using $A^\mu = (\Phi, \vec{A}\,)$ with the identifications $\vec{E} = -\,\partial_0 \vec{A} - \vec{\nabla}\Phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\begin{aligned} F^{i0} &= E^i \,, & F^{ij} &= -\varepsilon^{ijk} B^k \,, \\ \widetilde{F}^{i0} &= B^i \,, & \widetilde{F}^{ij} &= \varepsilon^{ijk} E^k \,. \end{aligned}$$

4. Stress Tensor. Using a known expression for $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma}$ the stress tensor can be written as

$$\Theta_\mu^{\;\nu} \;=\; F_\mu^{\;\sigma} F_\sigma^{\;\nu} \;-\; \tfrac{1}{4}\, \delta_\mu^{\;\nu} F_\rho^{\;\sigma} F_\sigma^{\;\rho} \;=\; \tfrac{1}{2} \left(F_\mu^{\;\sigma} F_\sigma^{\;\nu} \;+\; \widetilde{F}_\mu^{\;\sigma} \widetilde{F}_\sigma^{\;\nu} \right)$$

leading to expressions for energy density and momentum density in terms of fields $\vec{E}$ and $\vec{B}$. Conservation of energy and momentum for particles and fields in combination follows from

$$\partial^\mu\, \Theta_\mu^{\;\nu} \;=\; c^{-1} J^\mu F_\mu^{\;\nu} \,.$$

SYSTEMS OF UNITS FOR CLASSICAL ELECTRODYNAMICS

The electric field is defined in a way that relates to force per unit charge. Coulomb's law of force indicates that the electric field $\vec{E}$ at a location $\vec{r}$ relative to a stationary charge q is

$$k_e\,\vec{E} \;=\; \frac{1}{4\pi}\,\frac{q}{r^2}\,\frac{\vec{r}}{r}$$

where k_e is a constant multiplier of $\vec{E}$ that depends on the system of units employed. The charge is assumed to be isolated in free space.

Ampère's law and the Biot–Savart law may be re-written to express the magnetic induction field $\vec{B}$ at location $\vec{r}$ relative to an isolated charge q moving slowly at constant velocity $\vec{v}$

$$k_m\,\vec{B} \;=\; \frac{1}{4\pi}\,\frac{q}{r^2}\,\frac{\vec{v}}{c}\times\frac{\vec{r}}{r}$$

where k_m is a constant multiplier of $\vec{B}$ that depends on the system of units employed. The velocity of the non-accelerating charge is non-relativistic so that $|\vec{v}\,| \ll c$.

The Maxwell equations may be written in terms of the constant k_e associated with $\vec{E}$ and the constant k_m associated with $\vec{B}$, where ∂_0 refers to the operator $c^{-1}\partial/\partial t$,

$$\begin{aligned} k_e\vec{\nabla}\cdot\vec{E} &= \rho\,, & k_m\vec{\nabla}\times\vec{B} - k_e\,\partial_0\vec{E} &= \vec{J}/c\,, \\ k_m\vec{\nabla}\cdot\vec{B} &= 0\,, & k_e\vec{\nabla}\times\vec{E} + k_m\,\partial_0\vec{B} &= \vec{0}\,. \end{aligned}$$

The various systems of units in common use are summarised in the following table. The third edition of *Classical Electrodynamics* by J. D. Jackson uses the Gaussian system of units in its later sections.

System of units	k_e	k_m
Electrostatic (esu)	$1/4\pi$	$c/4\pi$
Electromagnetic (emu)	$1/4\pi c^2$	$4\pi c$
Gaussian	$1/4\pi$	$1/4\pi$
Heaviside-Lorentz	1	1
Rationalised MKSA	ϵ_0	$1/\mu_0 c$

The symbol c represents the limiting relativistic velocity and is likely to be the velocity of light. For all systems of units the continuity equation for charge density ρ and 3-current density $\vec{J}$ is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} \;=\; 0\,.$$