Is Light Heavy?

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Abstract

It is always valid to ask if we are justified in assuming a zero rest mass for the photon. We follow Proca's treatment of a massive vector field with external source and derive the Symmetric Stress Tensor, its Conservation Laws, and the Equations of Motion. We then use these to predict the effect a heavy photon would have on the Earth's Magnetic Field, following a method proposed by Schrödinger. Using information on the Earth's Magnetic field, we put an upper limit on the mass of the photon of ~4.10^{-48} g.

"… being heavy, I will bear the light …" proclaimed the melancholy hero in Shakespeare's Romeo and Juliet. Is light heavy? What evidence is there to support the assumption of Classical Electrodynamics that the photon has a zero rest mass? These are valid questions and we shall consider them here. First, we will assume that the photon has indeed a non-zero rest mass and derive its properties. Then, in section 2, following a method proposed by Schrödinger we will predict the effect a massive photon would have on the Earth's magnetic field and use recent data to place an upper limit on the photon's rest mass.

1. The Proca Lagrangian

In 1930 Proca considered a Lagrangian density for a massive vector field, in interaction with some external source J_{υ}. To calculate any observable effects of a heavy photon we will need to consider such a Lagrangian density. For completeness, we derive the Symmetric Energy-Momentum Stress Tensor (with its time-time and space-time components in terms of E and B) and the differential conservation laws it obeys, as well as the equations of motion which we will need for the next section.

(a) The Stress Tensor

The Proca Lagrangian density for a massive vector field in interaction with an external source is given as

$$L_p = \frac{-1}{16\pi} F_{\mu\upsilon} F^{\mu\upsilon} + \frac{m^2}{8\pi} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu$$

with $m = m_\gamma c/\hbar$, the Compton wave number of the photon, and m_γ the field mass.

Now, define the Energy-Momentum Stress Tensor as

$$T^{\mu\upsilon} = \left[\frac{\partial L_p}{\partial\left(\partial_\mu A_\sigma\right)}\right]\partial^\upsilon A_\sigma - g^{\mu\upsilon}L_p$$

so,

$$T^{\mu\upsilon} = \frac{-1}{4\pi}F^{\mu\sigma}\partial^\upsilon A_\sigma + \frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu$$

$$= \frac{1}{4\pi}g^{\beta\upsilon}F^{\mu\sigma}F_{\sigma\beta} - \frac{1}{4\pi}F^{\mu\sigma}\partial_\sigma A^\upsilon + \frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu$$

$$\begin{aligned} &= \frac{1}{4\pi}g^{\beta\upsilon}F^{\mu\sigma}F_{\sigma\beta} - \frac{1}{4\pi}\left[F^{\mu\sigma}\partial_\sigma A^\upsilon + A^\upsilon\partial_\sigma F^{\mu\sigma} + A^\upsilon\left(\frac{4\pi}{c}J^\mu - m^2A^\mu\right)\right] \\ &+ \frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu \end{aligned}$$

by equations of motion,

$$\begin{aligned} &= \frac{1}{4\pi}g^{\beta\upsilon}F^{\mu\sigma}F_{\sigma\beta} - \frac{1}{4\pi}A^\upsilon\left(\frac{4\pi}{c}J^\mu - m^2A^\mu\right) - \frac{1}{4\pi}\partial_\sigma\left(F^{\mu\sigma}A^\upsilon\right) \\ &+\frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu \end{aligned}$$

So, we may construct the Symmetric Energy Momentum Stress Tensor by defining

$$\Theta^{\mu\upsilon} = T^{\mu\upsilon} - T_D^{\mu\upsilon}$$

where

$$\begin{aligned} 2\,T_D^{\mu\upsilon} &= \frac{1}{4\pi}g^{\beta\upsilon}F^{\mu\sigma}F_{\sigma\beta} - \frac{1}{4\pi}A^\upsilon\left(\frac{4\pi}{c}J^\mu - m^2A^\mu\right) \\ &+\frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu \\ &-\left\{\frac{1}{4\pi}g^{\beta\upsilon}F^{\mu\sigma}F_{\sigma\beta} - \frac{1}{4\pi}A^\upsilon\left(\frac{4\pi}{c}J^\mu - m^2A^\mu\right) + \frac{1}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta\right. \\ &\left.-\frac{m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha + \frac{1}{c}g^{\mu\upsilon}J_\mu A^\mu - 2.\frac{1}{4\pi}\left(\partial_\sigma\left(F^{\mu\sigma}A^\upsilon\right)\right)\right\} \end{aligned}$$

So,

$$2\Theta^{\mu\upsilon} = \frac{1}{4\pi}g^{\beta\upsilon}F^{\upsilon\sigma}F_{\sigma\beta} + \frac{1}{4\pi}g^{\mu\beta}F^{\upsilon\sigma}F_{\sigma\beta} + \frac{2}{8\pi}g^{\mu\upsilon}F^{\sigma\beta}\partial_\sigma A_\beta - \frac{1}{4\pi}A^\upsilon\left(\frac{4\pi}{c}J^\mu - m^2A^\mu\right)$$

$$-\frac{1}{4\pi}A^\mu\left(\frac{4\pi}{c}J^\upsilon - m^2A^\upsilon\right) + \frac{1}{c}J^\upsilon A^\mu + \frac{1}{c}J^\mu A^\upsilon - \frac{2m^2}{8\pi}g^{\mu\upsilon}A_\alpha A^\alpha$$

$$= \frac{1}{4\pi}g^{\mu\iota}F_{\iota\sigma}F^{\sigma\upsilon} + \frac{1}{16\pi}g^{\mu\upsilon}F_{\sigma\beta}F^{\sigma\beta} + \frac{m^2}{4\pi}\left(A^\mu A^\upsilon - \frac{1}{2}g^{\mu\upsilon}A_\alpha A^\alpha\right)$$

So, finally, we have that

$$4\pi\Theta^{\mu\upsilon} = g^{\mu\iota}F_{\iota\sigma}F^{\sigma\upsilon} + \frac{1}{4}g^{\mu\upsilon}F_{\iota\sigma}F^{\iota\sigma} + m^2\left(A^\mu A^\upsilon - \frac{1}{2}g^{\mu\upsilon}A_\iota A^\iota\right)$$

(b) Equations of Motion

For the Proca Field we have the Euler-Lagrange equations of motion as

$$\partial_\mu \left(\frac{\partial L_p}{\partial \left(\partial_\mu A_\upsilon \right)} \right) - \frac{\partial L_p}{\partial A_\upsilon} = 0$$

Thus

$$\partial_\mu \left(\frac{\partial L_p}{\partial \left(\partial_\mu A_\upsilon \right)} \right) = \partial_\mu \left(\frac{\partial}{\partial \left(\partial_\mu A_\upsilon \right)} \left(\frac{-1}{16\pi} F_{\iota\sigma} F^{\iota\sigma} \right) \right)$$

$$= \partial_\mu \left(\frac{\partial}{\partial \left(\partial_\mu A_\upsilon \right)} \left(\frac{-1}{16\pi} \left(\partial_\iota A_\sigma - \partial_\sigma A_\iota \right) \left(\partial^\iota A^\sigma - \partial^\sigma A^\iota \right) \right) \right)$$

$$= \frac{-1}{16\pi} \partial_\mu \left(\frac{\partial}{\partial \left(\partial_\mu A_\upsilon \right)} \left(2 \left(\partial_\iota A_\sigma \right) \left(\partial^\iota A^\sigma \right) - 2 \left(\partial_\sigma A_\iota \right) \left(\partial^\sigma A^\iota \right) \right) \right)$$

$$= \frac{-1}{16\pi} \partial_\mu \left(2\partial^\mu A^\upsilon + 2\partial^\mu A^\upsilon - 2\partial^\upsilon A^\mu - 2\partial^\upsilon A^\mu \right)$$

$$= \frac{-1}{4\pi} \partial_\mu F^{\mu\upsilon}$$

Similarly,

$$\frac{\partial L_p}{\partial A_\upsilon} = \frac{\partial}{\partial A_\upsilon} \left(\frac{-1}{16\pi} F_{\mu\upsilon} F^{\mu\upsilon} + \frac{m^2}{8\pi} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu \right)$$

$$= \frac{\partial}{\partial A_\upsilon} \left(\frac{m^2}{8\pi} g^{\mu\upsilon} A_\mu A_\upsilon - \frac{1}{c} g^{\mu\upsilon} J_\mu A_\upsilon \right)$$

$$= \frac{m^2}{4\pi} g^{\mu\upsilon} A_\mu - \frac{1}{c} g^{\mu\upsilon} J_\mu$$

So,

$$\frac{-1}{4\pi} \partial_\mu F^{\mu\upsilon} - \frac{m^2}{4\pi} g^{\mu\upsilon} A_\mu + \frac{1}{c} g^{\mu\upsilon} J_\mu = 0$$

So, the Proca equations of motion for the massive field are given as,

$$\partial_\mu F^{\mu\upsilon} + m^2 A^\upsilon = \frac{4\pi}{c} J^\upsilon$$

(c) Conservation Laws

Look at the differential conservation laws for the massive vector field,

$$\partial_\mu \Theta^{\mu\upsilon} = \frac{1}{4\pi}\partial_\mu \left(g^{\mu\iota} F_{\iota\sigma} F^{\sigma\upsilon} + \frac{1}{4} g^{\mu\upsilon} F_{\iota\sigma} F^{\iota\sigma} + m^2 \left(A^\mu A^\upsilon - \frac{1}{2} g^{\mu\upsilon} A_\iota A^\iota \right) \right)$$

$$= \frac{1}{4\pi} \left(\partial^\iota \left(F_{\iota\sigma} \right) F^{\sigma\upsilon} + F_{\iota\sigma} \partial^\iota \left(F^{\sigma\upsilon} \right) + \frac{1}{2} F_{\iota\sigma} \partial^\upsilon \left(F^{\iota\sigma} \right) + m^2 \partial_\mu \left(A^\mu A^\upsilon \right) - \frac{1}{2} \partial^\upsilon \left(A_\iota A^\iota \right) \right)$$

$$= \frac{1}{4\pi} \left(\left(\frac{4\pi}{c} J_\sigma - m^2 A_\sigma \right) F^{\sigma\upsilon} + F_{\iota\sigma} \partial^\iota \left(F^{\sigma\upsilon} \right) + \frac{1}{2} F_{\iota\sigma} \partial^\upsilon \left(F^{\iota\sigma} \right) + m^2 \partial_\mu \left(A^\mu A^\upsilon \right) - \frac{1}{2} \partial^\upsilon \left(A_\iota A^\iota \right) \right)$$

by the equations of motion. Therefore,

$$4\pi \left(\partial_\mu \Theta^{\mu\upsilon} - \frac{1}{c} J_\sigma F^{\sigma\upsilon} \right) = F_{\iota\sigma} \partial^\iota \left(F^{\sigma\upsilon} \right) + \frac{1}{2} F_{\iota\sigma} \partial^\upsilon \left(F^{\iota\sigma} \right) + m^2 \partial_\mu \left(A^\mu A^\upsilon \right) - \frac{1}{2} \partial^\upsilon \left(A_\iota A^\iota \right) - m^2 A_\sigma F^{\sigma\upsilon}$$

Examine the term

$$F^{\mu\sigma} \partial_\mu F_{\sigma\nu} + \frac{1}{2} F^{\mu\sigma} \partial_\nu F_{\mu\sigma}$$

$$= \frac{1}{2} F_{\varphi\sigma} [(\partial^\mu F^{\varphi\sigma} + \partial^\varphi F^{\sigma\nu}) + \partial^\varphi F^{\sigma\nu}]$$

now,

$$\partial_\mu \widetilde{F^{\mu\nu}} = 0$$

therefore

$$\partial_\nu (3\varepsilon^{\mu\nu\varphi\sigma} F_{\varphi\sigma}) = 0$$

which implies

$$6(2\partial_\nu F_{\varphi\mu} + \partial_\varphi F_{\mu\nu} + \partial_\mu F_{\nu\varphi}) = 0$$

so

$$\frac{1}{2} F_{\varphi\sigma} [(\partial^\mu F^{\varphi\sigma} + \partial^\varphi F^{\sigma\nu}) + \partial^\varphi F^{\sigma\nu}]$$

$$= \frac{1}{2} F_{\varphi\sigma} [\partial^\sigma F^{\varphi\nu} + \partial^\varphi F^{\sigma\nu}] = 0$$

which gives finally:

$$4a(\partial_\mu \Theta^{\mu\nu} \frac{1}{c} J_\sigma F^{\sigma\nu}) = m^2 (\partial_\mu (A^\mu A^\nu) - \frac{1}{2} \partial^\nu (A^\varphi A_\varphi) - A_\sigma F^{\sigma\nu})$$

and in particular if we employ the Lorentz gauge we have that:

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} J_\varphi F^{\varphi\nu}$$

as for the massless electromagnetic field.

(d) Components

Here we give the time-time components of the Proca symmetric stress momentum energy tensor

$$
\begin{aligned}
8\pi\Theta^{00} &= 2F_{0i}F^{i0} + \frac{1}{2}F_{0i}F^{0i} + \frac{1}{2}F_{0i}F^{i0} + \frac{1}{2}F_{ij}F^{ij} + 2m^2(A^0A^0 - A_iA^i) \\
&= 2E^iE^i - E^iE^i + \frac{1}{2}(-\varepsilon^{ijk}B^k)(-\varepsilon^{ijl}B^l) + m^2(A^0A^0 - A^iA^i) \\
&= \vec{E}^2 + \vec{B}^2 + m^2(A^0A^0 - A^iA^i)
\end{aligned}
$$

Here, we exhibit the space-time components of the Proca tensor as,

$$
\begin{aligned}
8\pi\Theta^{i0} &= g^{i0}F_{0l}F^{l0} + g^{(ii)}F_{il}F^{l0} + m^2\left(A^iA^0 - \tfrac{1}{2}\,g^{l0}A_iA^i\right) \\
&= g^{(ii)}F_{ij}F^{j0} + m^2(A^iA^0) \\
&= \varepsilon^{ijk}E^iB^k + m^2A^iA^0 \\
&= (\vec{E}\times\vec{B})^i + m^2A^iA^0
\end{aligned}
$$

2. On a Geomagnetic Limit for the Photon Rest Mass

In 1943 Schrödinger proposed a method of estimating the mass of a photon. This, he claimed, would have a measurable effect on the Earth's magnetic field, and using sparse and innaccurate data puts forward a conservative estimate. We improve his method using the machinery we developed in the previous section, and propose a significantly smaller number for the mass of the photon with the aid of more accurate data from recent satellite and surface observations.

(a) Massive Vector Potential

The Proca equations of motion are:

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c}J_\alpha$$

where

$$F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta$$

Imposing the Lorentz guage condition

$$\partial^\beta A_\beta = 0$$

this implies that

$$(\partial^\beta\partial_\beta + \mu^2)A_\alpha = \frac{4\pi}{c}J_\alpha$$

For a steady-state distribution of current this becomes

$$(\nabla^2 - \mu^2)\vec{A} = -\frac{4\pi}{c}\vec{J}$$

We require a Green's Function satisfying

$$(\nabla^2 - \mu^2)G(r) = -4\pi\delta(r)$$

Consider r$\neq$0 :

$$(\nabla^2 - \mu^2)G(r) = 0$$

as G = G(r) we can write

$$\nabla^2 G = \frac{1}{r}\frac{d^2}{dr^2}(rG)$$

thus :

$$\frac{d^2}{dr^2}(rG) = \mu(rG)$$

$$rG = K \exp \pm \mu r$$

As ϕ goes to as r$\rightarrow \infty$ we take

$$rG = K \exp -\frac{\mu r}{r}$$

the (spherically symmetric) Yukawa potential. We require $K = -\frac{1}{4\pi}$ so that :

$$(\nabla^2 - \mu^2)G(r) = -4\pi\delta(r)$$

for all r and:

$$G(\vec{x}, \vec{x'}) = -\frac{1}{4\pi}\frac{\exp -\mu \mid \vec{x} - \vec{x'} \mid}{\mid \vec{x} - \vec{x'} \mid}$$

thus

$$\vec{A} = -\frac{4\pi}{c}\int G(\vec{x}, \vec{x'})\vec{J}(\vec{x'})d^3x'$$

$$= \frac{1}{c}\int \frac{\exp -\mu \mid \vec{x} - \vec{x'} \mid}{\mid \vec{x} - \vec{x'} \mid}\vec{J}(\vec{x'})d^3x'$$

If there is a magnetisation $\vec{M} = \vec{m}f(\vec{x})$ where $\vec{m}$ is a fixed vector and $f(\vec{x})$ is a localised scalar function then

$$\vec{J} = c\nabla \times \vec{m}f(x)$$

noting that in a steady state distribution of current the Lorentz guage becomes the Coulomb guage, so

$$\nabla \cdot \vec{A} = 0$$

and we obtain

$$\vec{A} = -\vec{m} \times \nabla \int f(x)\frac{\exp -\mu \mid \vec{x} - \vec{x'} \mid}{\mid \vec{x} - \vec{x'} \mid}d^3x'$$

(b) Magnetic field

If the magnetic dipole is a point dipole at the origin then $f(x) = \delta(x)$ and

$$\vec{A} = -\vec{m} \times \nabla \frac{e^{-\mu r}}{r}$$

with $\vec{B} = \nabla \times \vec{A}$. We have

$$\nabla \frac{e^{-\mu r}}{r} = -\frac{e^{-\mu r}}{r^3}(\mu r + 1)\vec{r} \equiv R\vec{r}$$

Thus

$$\vec{B} = \nabla \times ((R\vec{r}) \times \vec{m})$$

$$= (\vec{m} \cdot \nabla)(R\vec{r}) - \vec{m}\,(\nabla \cdot (R\vec{r}))$$

where we have used the identity

$$\nabla \times \left(\vec{F} \times \vec{G}\right) = \left(\vec{G} \cdot \nabla\right)\vec{F} - \vec{G}\left(\nabla \cdot \vec{F}\right) - \left(\vec{F} \cdot \nabla\right)\vec{G} + \vec{F}\left(\nabla \cdot \vec{G}\right)$$

and the fact that $\vec{m}$ is a constant vector. Consider now
$(\vec{m}.\nabla)(\phi\vec{r}) \equiv m_a \frac{\partial}{\partial x_a}(\phi x_b \vec{e^b})$

using Einstein's summation convention (a=1,2,3) with $\vec{e^b}$ the usual orthonormal basis vectors. Thus
$(\vec{m}.\nabla)(\phi\vec{r}) \equiv m_a\left(\frac{\partial\phi}{\partial x_a} x_b \vec{e^b} + \phi\delta_b^a \vec{e^b}\right)$
$\equiv (\vec{m}.\nabla\phi)\vec{r} + \vec{m}\phi$

$$= 3\frac{e^{-\mu r}}{r^3}\left(1 + \mu r + \frac{\mu^2 r^2}{3}\right)(\hat{r}.\vec{m})\,\hat{r} - \frac{e^{-\mu r}}{r^3}(\mu r + 1)\,\vec{m}$$

Now

$$\nabla.(\phi\vec{r}) = \phi\,(\nabla.\vec{r}) + (\nabla\phi).\vec{r} = 3\phi + (\nabla\phi).\vec{r} = \frac{e^{-\mu r}}{r^3}\left(\mu^2 r^2\right)$$

So we obtain,
$\vec{B} = [3\,(\hat{r}.\vec{m})\,\hat{r} - \vec{m}]\left(1 + \mu r + \frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r} - \frac{2}{3}\mu^2 \frac{e^{-\mu r}}{r}\vec{m}$

(c) Photon mass limit

J. D. Jackson tells us that

$$\frac{H_{ext}}{H_{AD}} \sim 4 \times 10^{-3}$$

(upper limit) on the surface of the earth (r=R). Thus

$$\frac{\frac{2}{3}(\mu R)^2}{1 + \mu R + \frac{1}{3}(\mu R)^2} = 4 \times 10^{-3}$$

yielding,

$$\mu R = 0.0806$$

This gives us a lower limit on $\mu^{-1}$of

$$\mu^{-1} = 12.407$$

earth radii. Schrödinger, in his study, calls this the "characteristic length". It is, of course, the reciprocal Compton wavelength of a photon

$$\mu = \frac{c}{\hbar} m$$

Thus, an upper limit of the rest mass of the photon is

$$m \sim 4 \times 10^{-48} \text{ g}.$$

[A review of past and current studies of possible long-distance, low-frequency deviations from Maxwell electrodynamics and Einstein gravity has appeared in [4], NHB.]

References

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