

1. Consider n identical point masses on a straight line connected by $n - 1$ identical springs.

(a) Given that the Euler-Lagrange equations and Lagrangian for this system are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}_i} \right) = \frac{\partial L}{\partial \varphi_i}, \quad L = \sum_{i=1}^n \frac{1}{2} m \dot{\varphi}_i^2 - \sum_{i=1}^{n-1} \frac{1}{2} k (\varphi_{i+1} - \varphi_i)^2,$$

where φ_i , $i = 1, \dots, n$, is the displacement from equilibrium of the i^{th} point mass of mass m , and where $\dot{\varphi}_i = d\varphi_i/dt$, find the equations of motion of the system [4 marks].

(b) Keeping the linear mass density constant as the mass m of each point mass tends to zero, in concert with the distance between the masses tending to zero, derive the equations of motion for the above system in the continuum limit [9 marks].

(c) Show that the equations of motion follow from an appropriate Lagrangian density and corresponding Euler-Lagrange equations [4 marks].

2. Outline the general form of terms that may contribute to a Lagrangian describing the relativistic motion of a particle of charge e and momentum $p^\mu = (\mathcal{E}/c, \vec{p})$ interacting with an electromagnetic four-potential A_μ [2 marks].

(a) Given the following Lagrangian for a particle of rest-mass m located at point x^μ

$$L = -m c^2 - \frac{e}{c} A_\mu \frac{dx^\mu}{d\tau}$$

find the relativistic equation of motion by using Hamilton's variational principle for the action $S = \int L d\tau$ with squared invariant time increment $d\tau^2 = dt^2 - d\vec{x}^2/c^2$ [7 marks].

(b) Write down the formulæ relating the electric and magnetic fields \vec{E} and \vec{B} to the four potential $A^\mu \equiv (\Phi, \vec{A})$ [2 marks].

(c) From the relativistic equation of motion obtained above derive the equations

$$\frac{d\mathcal{E}}{dt} = e \vec{E} \cdot \vec{v}, \quad \frac{d\vec{p}}{dt} = e \vec{E} + \frac{e}{c} \vec{v} \times \vec{B} \quad [6 \text{ marks}].$$

3. Consider a Lagrangian density written in terms of a four current density $J^\mu = (c\rho, \vec{J})$ and field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (∂_μ represents $\partial/\partial x^\mu$) in Heaviside-Lorentz units

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} A_\mu J^\mu.$$

- (a) Use the Euler-Lagrange equations to derive the inhomogeneous Maxwell equation

$$\partial_\mu F^{\mu\nu} = c^{-1} J^\nu \quad [6 \text{ marks}].$$

- (b) Use the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ to prove that the homogeneous equation is

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad [4 \text{ marks}].$$

- (c) Relate components of the dual tensor $\tilde{F}^{\mu\nu}$ to elements of the electromagnetic fields \vec{E} and \vec{B} [3 marks].

- (d) Express the above four dimensional forms of the inhomogeneous and homogeneous Maxwell-Hertz equations in terms of the three-vectors \vec{E} and \vec{B} [4 marks].

4. Find an explicit form for the canonical stress tensor of a free electromagnetic field without sources in Heaviside-Lorentz units

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L}, \quad \text{given } \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad [4 \text{ marks}].$$

- (a) Show by direct calculation that the canonical stress tensor is conserved

$$\partial_\mu T^{\mu\nu} = 0 \quad [2 \text{ marks}].$$

- (b) Explain why the canonical stress tensor $T^{\mu\nu}$ is not gauge invariant [2 marks].

- (c) Indicate in detail how to construct, from the tensor $T^{\mu\nu}$, the gauge invariant tensor

$$\Theta^{\mu\nu} = g^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \quad [4 \text{ marks}].$$

- (d) Demonstrate that the gauge invariant tensor $\Theta^{\mu\nu}$ is (i) conserved, (ii) symmetric, and (iii) traceless [5 marks].

5. Discuss the Fourier transform of the Dirac delta function and find an integral representation of the delta function [2 marks].

(a) Show that one form of the Fourier transform of the Green function for d'Alembert's wave equation may be written as follows, in a metric where $k_\mu k^\mu = k_0^2 - \vec{k}^2$

$$\tilde{D}(k) = -\frac{1}{4\pi^2 k_\mu k^\mu} \quad [4 \text{ marks}].$$

(b) Evaluate the retarded Green function for the wave equation and verify that it may be written in the form

$$D_{\text{ret}}(z^\mu) = \frac{\theta(z^0) \delta(z^0 - |\vec{z}|)}{4\pi |\vec{z}|} \quad [9 \text{ marks}]$$

where $z^\mu = x^\mu - x'^\mu$ has components (z^0, \vec{z}) .

(c) Without proof, write down the advanced Green function for the wave equation in covariant and non-covariant forms [2 marks].

6. Using the Green function calculated in the previous question, write down a solution for the following wave equation in terms of an integration over a given four current $J^\mu(x)$,

$$\frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x_\nu} A^\mu = \frac{1}{c} J^\mu \quad [2 \text{ marks}].$$

(a) Derive the following formula in Heaviside-Lorentz units for the four vector potential A^μ at the space-time point x^μ

$$A^\mu = \frac{e}{4\pi} \frac{V^\mu(\tau_0)}{V_\sigma(\tau_0) [x^\sigma - x_e^\sigma(\tau_0)]} \quad [10 \text{ marks}].$$

for a charge moving with 4-velocity $V^\mu = dx_e^\mu/d\tau$ at location $x_e^\mu(\tau)$, with τ being the proper time.

(b) Define the retarded time τ_0 appearing in the expression for the 4-vector potential at space-time event x^μ [2 marks].

(c) Find expressions describing the Liénard-Wiechert potentials for a moving charged particle in terms of $\vec{R} \equiv \vec{x} - \vec{x}_e(t)$ and $\vec{v} \equiv d\vec{x}_e/dt$ [3 marks].

7. From the formula for the electromagnetic potential in the previous question, obtain an expression for the field $F_{\mu\nu}$ due to a moving particle of charge e whose space-time location at proper time τ is given by $x_e^\mu(\tau)$ [8 marks].

(a) Show that the non-radiative velocity fields $F_{\text{vel}}^{\mu\nu}$ tend to the electromagnetic fields of a static charge in the limit of zero 3-velocity [4 marks].

(b) Demonstrate that the radiative fields may be written in Heaviside-Lorentz units with retarded time as

$$\vec{E}_{\text{rad}} = \frac{e}{4\pi c} \cdot \frac{\vec{n} \times \left[(\vec{n} - \vec{\beta}) \times \vec{\alpha} \right]}{(1 - \vec{\beta} \cdot \vec{n})^3 R}, \quad \vec{B}_{\text{rad}} = \vec{n} \times \vec{E}_{\text{rad}},$$

where $R\vec{n} = \vec{R} = \vec{x} - \vec{x}_e$, $R = |\vec{R}|$, where $c\vec{\beta} = -d\vec{R}/dt$ is the 3-velocity and where $c\vec{\alpha} = cd\vec{\beta}/dt$ is the 3-acceleration of the charged particle [5 marks].

8. A non-relativistic, but accelerating, particle of mass m and charge e in Heaviside-Lorentz units radiates energy.

(a) By requiring that the kinetic energy gain be much greater than the energy lost through radiation, show that radiative effects do not modify the motion of the particle for uniform accelerations in a straight line acting over times

$$T \gg e^2 / (3\pi m c^3) \quad [6 \text{ marks}].$$

(b) Repeat the analysis for the case of the same particle moving in a circle at a constant non-relativistic speed, taking time T to complete a full turn. Verify that the criterion is independent of the radius of the circular path followed [6 marks].

(c) Show that the radiation energy lost in the case of periodic motion at non-relativistic velocity \vec{v} may be represented in Heaviside-Lorentz units by a reaction force

$$\vec{F}_{\text{rad}} = \frac{e^2}{6\pi c^3} \frac{d^2\vec{v}}{dt^2} \quad [5 \text{ marks}].$$