Nonhomogeneous ODEs 1S2 JF Natural Sciences

1. Explain what is meant by the term Ordinary Differential Equation (ODE).

2. Explain the term Linear ODE of second order with constant coefficients.

3. Find the general solution of the following linear homogeneous ODE

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$$

4. Obtain a particular solution of the nonhomogeneous ODE

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 5e^{4t}$$

5. Determine the general solution of the nonhomogeneous ODE

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 8 - 6t$$

6. Seek a particular solution of the nonhomogeneous ODE

$$\frac{d^2y}{dt^2} + 4y = 10\sin(3t) - 12\cos(4t)$$

7. For each of the above three nonhomogeneous ODEs calculate the solution that satisfies the initial conditions

$$y = 0 \text{ at } t = 0, \text{ and } \frac{dy}{dt} = 0 \text{ at } t = 0.$$

Background Explanations for Non Homogeneous ODEs

- The general solution of the following linear homogeneous ODE written with y' representing the first derivative and y'' representing the second derivative, $y'' - 6y' + 5y = 0$, is found from trial solutions of the form $y = e^{\lambda t}$ leading, on substitution and division by the nonzero $e^{\lambda t}$ to the quadratic equation
$$\lambda^2 - 6\lambda + 5 = 0$$
with roots $\lambda_1 = 1$ and $\lambda_2 = 5$. It may be shown that the general solution y_h of the homogeneous ODE, and this may be checked by substitution, involves a linear combination of trial solutions, weighted with arbitrary constants c_1 and c_2,
$$y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{t} + c_2 e^{5t}.$$

- A particular solution of a nonhomogeneous ODE with polynomial right side
$$y'' - 4y' - 5y = 10t - 7$$
is found from trial solutions of polynomial form, often of a similar degree,
$$y_p = bt + c$$
whose undetermined constant coefficients b and c may be found by noting
$$y' = b \quad \text{and} \quad y'' = 0$$
before substituting y, y' and y'' in the ODE leading to the identification
$$0 - 4b - 5(bt + c) = 10t - 7.$$
For agreement at all values of t, the constant and t terms must match; so
$$b = -2 \quad \text{and} \quad c = 3$$
and the particular solution to the ODE is therefore $y_p = 3 - 2t$.

- A particular solution of a nonhomogeneous ODE with functions such as
$$y'' - 6y' + 5y = \sin(2t) + e^{3t}$$
is similarly obtained by finding the coefficients a, b, c when substituting a trial solution necessarily including sines and cosines with the same argument
$$y_p = a\cos(2t) + b\sin(2t) + ce^{3t}.$$