SCHOOL OF MATHEMATICS TRINITY COLLEGE DUBLIN

EIGENVECTORS 1S2 TUTORIAL JF SCIENCES

Consider the following two-by-two matrix A and the three-by-three matrix B

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 8 & 0 & 0 \\ 10 & 1 & 1 \end{bmatrix}.$$

1. Is the above matrix A a symmetric matrix? Explain the answer you provide.
2. Determine the eigenvalues of the matrix A. Are the eigenvalues distinct?
3. Find the eigenvector of the matrix A related to the least eigenvalue of A and the eigenvector of the matrix A related to the greatest eigenvalue of A.
4. Are the eigenvectors V and V' of the matrix A orthogonal; is it true that

$$V^T V' = 0\,?$$

5. Obtain the eigenvalues of the 3×3 matrix B. How many are there?
6. Find the eigenvectors of the matrix B for each of the eigenvalues of B.
7. Verify that each eigenvector V of B and its corresponding eigenvalue λ obeys the matrix equation

$$B V = \lambda V\,.$$

8. Consider an invertible matrix C. Given the eigenvalues of the matrix C show how to obtain the eigenvalues of the inverse of the matrix C, without inverting the matrix C. Is the set of eigenvectors of the matrix C the same as the set of eigenvectors of the inverse of the matrix C?

Revision challenge: show that the eigenvectors associated with distinct eigenvalues of a general symmetric matrix are mutually orthogonal.

See Section 4.4 of *Contemporary Linear Algebra* by Howard Anton and Robert C. Busby (Wiley), or the end of Section 2.3 of *Elementary Linear Algebra & Applications* by Howard Anton and Chris Rorres (Wiley).

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A BACKGROUND ITEM FOR EIGENVECTORS OF A MATRIX

An eigenvector example for substitution rates among the purine and pyrimidine bases of DNA generalises readily to other biomolecule rates. The bases of DNA with a double-ring structure, Adenine and Guanine, are called purines (denoted R). The bases with a single-ring structure, Thymine and Cytosine, are pyrimidines (Y). A large base and a small base pair together, A–T, G–C, in the DNA double helix.

Averaged over some species, the proportion of purines (r%) and pyrimidines (y%) varies steadily as time proceeds over millions of years in such a way that, after a time period T, the values r and y, elements of a column vector X, change to

$$M\,X \quad = \quad \begin{bmatrix} 0.91 & 0.11 \\ 0.09 & 0.89 \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix} .$$

Q. Given that the values of r and y are 35% and 65% respectively at the current time of $t = 0$, find a formula for values of r and y at a general time $t = nT$ for integer n. Evaluate the proportions for time values $n = -2,\ -1,\ 1,\ 2,$ (and ∞).

Ans. Eigenvalues of the probability matrix M (columns sum to one) are given by

$$\det(M \; - \; \lambda\, I) \quad = \quad \begin{vmatrix} 0.91 - \lambda & 0.11 \\ 0.09 & 0.89 - \lambda \end{vmatrix} \quad = \quad \lambda^2 \, - 1.8\, \lambda \, + \, 0.8 \quad = \quad 0$$

leading to $\lambda = 0.8$ and $\lambda' = 1$. For each eigenvalue the related eigenvector is obtained from the defining equation $M\,V \; = \; \lambda\, V$, that is, $(M \; - \; \lambda\, I)\, V \; = \; 0$, or

$$\begin{bmatrix} 0.91 - \lambda & 0.11 \\ 0.09 & 0.89 - \lambda \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \quad = \quad 0\,.$$

Writing out this matrix equation in component form provides the eigenvectors V and V' of the matrix M corresponding to the eigenvalues $\lambda = 0.8$ and $\lambda' = 1$,

$$V = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad V' = \begin{bmatrix} u' \\ w' \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}.$$

Multiplying an eigenvector repeatedly by the matrix M yields for an integer n

$$M\,V \; = \; \lambda\, V\,, \qquad M^2\,V \; = \; \lambda^2\, V\,, \qquad \cdots\,, \qquad M^n\,V \; = \; \lambda^n\, V\,,$$

with similar equations for V' and λ'. If the linear form with constants k and k' in

$$M^n\,(k\,V \; + \; k'\,V') \quad = \quad k\,\lambda^n\,V \; + \; k'\,\lambda'^n\,V'$$

on the left can be matched to the values 35% and 65% at the time $t \; = \; 0$ by suitably choosing k and k', then the right side provides the solution required. Writing out the matrix equation $k\,V \; + \; k'\,V' \; = \; (35,\, 65)^T$ in component form readily leads to $k \; = \; 20$ and $k' \; = \; 5$, indicating that the required solution is

$$M^n \begin{bmatrix} 35 \\ 65 \end{bmatrix} = 20 \left(\frac{4}{5}\right)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 11 \\ 9 \end{bmatrix}; \begin{bmatrix} 23.75 \\ 76.25 \end{bmatrix}, \begin{bmatrix} 30 \\ 70 \end{bmatrix}; \begin{bmatrix} 39 \\ 61 \end{bmatrix}, \begin{bmatrix} 42.2 \\ 57.8 \end{bmatrix}$$